Reality Properties of Conjugacy Classes in Algebraic Groups

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Definition and Examples of Algebraic groups

Let K be an algebraically closed field.

Definition (Algebraic Group). *An algebraic group G over a field K is an algebraic variety defined over K which is also a group such that the maps defining group structure*

$$\mu\colon G \times G \to G, \mu(x, y) = xy$$

and

$$i\colon G \to G, i(x) = x^{-1}$$

are morphisms of varieties.

The group GL_n, SL_n, D_n (non-singular diagonal matrices), T_n (upper triangular matrices in GL_n), U_n (unipotent upper triangular matrices), O_n, SO_n, Sp_n, elliptic curves etc. are examples of algebraic groups.

An algebraic group G is called a **linear algebraic group** if the underlying variety of G is affine. Such groups can be embedded in GL_n for some n, hence the name. In what follows algebraic group will always refer to linear algebraic group.

Let k be a field. An algebraic group G is said to be defined over k if the underlying variety of G is defined over k. The notation $G(k)$ will denote the k points of G.

A Question About Reality of Elements

Let G be an algebraic group defined over k. An element $t \in G(k)$ is called **real** if there exists $g \in G(k)$ such that $gtg^{-1} = t^{-1}$.

Our question is when an element in $G(k)$ real?

This talk is about determining real elements (semisimple, unipotent or general elements) in algebraic groups and studying its structure.

I assume characteristic of $k \neq 2$, now onwards.

An element $t \in G$ is called an **involution** if $t^2 = 1$. Involutions play an important role in our investigation.

Strongly Real Elements

Definition (Strongly Real). *An element in G is called* **strongly real** *if it is a product of two involutions in G.*

Note that a strongly k-real element in $G(k)$ is always k-real in $G(k)$.
For if $t = \tau_1\tau_2$ with $\tau_i^2 = 1$ then

$$\tau_1.t.\tau_1^{-1} = \tau_1.\tau_1\tau_2.\tau_1 = \tau_2\tau_1 = \tau_2^{-1}\tau_1^{-1} = t^{-1}.$$

Conversely, a real element $t \in G(k)$ is strongly k-real if and only if there exists a conjugating element $\tau \in G(k)$ which is an involution, i.e., there exists $\tau \in G(k)$ with $\tau^2 = 1$ such that $\tau t \tau^{-1} = t^{-1}$. In that case, $t = \tau.\tau t$.

Let G be an algebraic group (semisimple) defined over k.

When a real element in $G(k)$ strongly real in $G(k)$?

Plan of the Talk

Part-I Real elements in some classical groups and in the groups of type G_2

Part-II Reality in linear algebraic groups

Part-III Groups of type G_2

Part-IV Conclusion and some questions

Part-I

Real Elements in Classical Groups and in G_2

The Groups GL_n and SL_n

Wonenburger (1966) proved that an element of $GL_n(k)$ is real if and only if it is strongly real in $GL_n(k)$.

Ellers (1977) showed that this result does not generalise to matrix algebras over division rings.

We have looked into the structure of real elements in $SL_n(k)$.

Theorem. *Let V be a vector space of dimension n over k. Let $t \in SL(V)$. Suppose $n \not\equiv 2 \pmod 4$. Then, t is real in $SL(V)$ if and only if t is strongly real in $SL(V)$.*

The Groups of type A_1

Any group of type A_1 over k is isomorphic to $SL_1(Q)$ for some Q, a quaternion algebra over k. That is, it is a form of SL_2 defined over k.

Let $Q = \left(\frac{a,b}{k}\right)$ be a quaternion algebra over k. That is, Q has a basis $\{1, i, j, ij\}$ with

$$i^2 = a, j^2 = b, ij = -ji.$$

It is a central simple algebra over k of degree 2 with norm defined by

$$N(x_0 1 + x_1 i + x_2 j + x_3 ij) = x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2.$$

We denote the set of norm 1 elements of Q by $SL_1(Q)$. We remark that $M_2(k)$ is a quaternion algebra with norm form given by determinant.

Proposition (A). *Let $t \in SL_2(k)$ be a real semisimple element. Then there exists $g \in SL_2(k)$ with $g^2 = -I$ such that $gtg^{-1} = t^{-1}$. In particular if t is a real semisimple element in $PSL_2(k)$ then t is strongly real.*

Proposition (B). *Let $G = PSL_1(Q)$ and $t \in G$ be a semisimple element. Then, t is real in G if and only if t is strongly real.*

Example : Let $\mathbb{H}$ be the quaternion division algebra over $\mathbb{R}$. Then $jij^{-1} = i^{-1}$, i.e., i is a real element but i is not a product of two involutions (only involutions are ± 1) whereas j is an involution in $PSL_1(\mathbb{H})$.

Proof of (A): Over $\bar{k}$ the element t is conjugate to the matrix $t_0 = \mathrm{diag}\{\alpha, \alpha^{-1}\}$ for some $\alpha \in \bar{k}$. If t is central then t is either I or $-I$ otherwise t is regular and $\alpha^2 \neq 1$. We write $n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and then $nt_0n^{-1} = t_0^{-1}$ where $n^2 = -1$ and $n \in SL_2(\bar{k})$. In fact n conjugates every element of the torus $T = \{\mathrm{diag}\{\alpha, \alpha^{-1}\} \mid \alpha \in \bar{k}\}$ to its inverse. Hence there exist $h \in SL_2(\bar{k})$ such that $hth^{-1} = t^{-1}$ and $h^2 = -I$. As t is real in $SL_2(k)$ there exists $g \in SL_2(k)$ such that $gtg^{-1} = t^{-1}$. Then $g \in h\mathcal{Z}_{GL_2(\bar{k})}(t)$. We note that as t is regular we have $\mathcal{Z}_{GL_2(\bar{k})}(t) = T$, a maximal torus. We write $g = hx$ where $x \in T$. We check that $g^2 = -I$ and this proves the required result. $\square$

Orthogonal Groups

Let k be a field (of char $\neq 2$) and V be a vector space over k of dimension n. Let Q be a non-degenerate quadratic form on V and B be the corresponding bilinear form on V. Let $O(Q) = \{t \in \mathrm{End}(V) \mid B(t(x), t(y)) = B(x, y)\}$ be the orthogonal group.

Then, it was proved by Wonenburger (1966) that every element of $O(Q)$ is a product of two involutions hence strongly real. Knuppel and Nielsen (1987) proved that if $n \not\equiv 2 \pmod 4$ then every element of $SO(Q)$ is a product of two involutions in $SO(Q)$. They also proved that any element of $SO(Q)$ (for any n) is a product of three involutions.

However we prove,

Theorem. *Let $t \in SO(Q)$ be a semisimple element. Then, t is real in $SO(Q)$ if and only if t is strongly real in $SO(Q)$.*

Symplectic Groups

Let k be a field (of char $\neq 2$) and V be a vector space over k of dimension $2n$. Let B be a skew-symmetric bilinear form on V. We denote

$$Sp(V,B) = \{t \in \mathrm{End}(V) \mid B(t(x),t(y)) = B(x,y)\}$$

and

$$ESp(V,B) = \{t \in \mathrm{End}(V) \mid B(t(x),t(y)) = \pm B(x,y)\}.$$

The group of similitude is denoted by

$$GSp(V,B) = \{t \in \mathrm{End}(V) \mid B(t(x),t(y)) = \mu(t)B(x,y), \mu(t) \in k^*\}$$

where $\mu(t)$ is similitude factor.

The elements $t \in ESp(V,B)$ which satisfy $B(t(x),t(y)) = -B(x,y)$ are called skew-symplectic.

Wonenburger (1966) proved that every element of $Sp(V, B)$ is a product of two skew-symplectic involutions.

Theorem. *Let $t \in PSp(2n, k)$ be a real, semisimple element. Then t is strongly real.*

Recently Vinroot (2004) analysed the group $GSp(2n, k)$ and proved following extension of Wonenburger's result. Let $g \in GSp(2n, k)$ with similitude factor $\mu(g) = \beta$. Then $g = t_1 t_2$, where t_1 is a skew-symplectic involution and t_2 is such that $\mu(t_2) = -\beta$ with $t_2^2 = \beta I$.

Unitary Groups

Let K be a quadratic extension of k. Let V be an n-dimensional vector space over K with hermitian form h. Then we have,

Theorem. *Let (V, h) be a hermitian space over K. Let $t \in U(V, h)$ be a semisimple element. Then, t is real in $U(V, h)$ if and only if it is strongly real.*

Theorem. *Let $t \in SU(V, h)$ be a semisimple element. Suppose $n \not\equiv 2 \pmod 4$. Then, t is real in $SU(V, h)$ if and only if it is strongly real.*

Groups of type G_2

It is known that for a group G of type G_2 over k, there exists an octonion algebra $\mathfrak{C}$ over k, unique up to a k-isomorphism, such that $G \cong \mathbf{Aut}(\mathfrak{C})$, the group of k-algebra automorphisms of $\mathfrak{C}$.

Octonion algebras (also called Cayley algebras) are 8-dimensional non-commutative, non-associative algebras obtained by doubling a quaternion algebra. Jacobson (1958) studied this group and some of its subgroups and proved that every element of $\mathrm{Aut}(\mathfrak{C})$ is a product of involutions. Wonenburger (1969) proved that every element is a product of three involutions.

We determine real elements in these groups and prove that,

Theorem. *In addition, if $char(k) \neq 3$, every unipotent element in $G(k)$ is strongly real in $G(k)$.*

For a general element in $G(k)$, we prove,

Theorem. *Let $char(k) \neq 2, 3$. Then, an element g is real in $G(k)$ if and only if it is strongly real in $G(k)$.*

Over finite fields every unipotent element as well as every semisimple element is a product of two involutions hence real. Though there are elements which are not real.

Proof in the case of $SO(Q)$

Lemma (A). *Let $t \in SO(Q)$ where $\dim(V) \equiv 2 \pmod 4$. Let t be a semisimple element which has only two distinct eigenvalues λ and λ^{-1} (hence $\lambda \neq \pm 1$) over $\bar{k}$. Then t is not real in $SO(Q)$.*

Proof : We prove that the element t is not real over $\bar{k}$. Let $\dim(V) = 2m$ where m is odd. The element t over $\bar{k}$ is conjugate to $A = \mathrm{diag}(\underbrace{\lambda, \ldots, \lambda}_{m}, \underbrace{\lambda^{-1}, \ldots, \lambda^{-1}}_{m})$ with $\lambda \neq \pm 1$ in $SO(J)$ where J is the matrix of the quadratic form over $\bar{k}$ given by

$$J = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \text{ where } S = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}, \text{ an } m \times m \text{ matrix.}$$

Now suppose A is real in $SO(J)$, i.e., there exists $T \in SO(J)$ such that $TAT^{-1} = A^{-1}$. Then T maps the λ-eigen subspace of A to the λ^{-1}-eigen subspace of A and vice-versa. Hence T has the following form:

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

for $m \times m$ matrices B and C. Since T is orthogonal, it satisfies ${}^tTJT = J$, which gives ${}^tBSC = S$. That is, $\det(B)\det(C) = 1$. Hence

$$\det(T) = (-1)^m \det(B)\det(C) = -\det(B)\det(C) = -1$$

since m is odd. This contradicts that $T \in SO(J)$. Hence A is not real in $SO(J)$ and hence t is not real in $SO(Q)$. $\square$

Lemma (B). *Let* $\dim(V) \equiv 0 \pmod 4$ *and* $t \in SO(Q)$ *be semisimple. Suppose* t *has only two distinct eigenvalues* λ *and* λ^{-1}*(hence* $\lambda \neq \pm 1$*) over* $\bar{k}$*. Then, any element* $g \in O(Q)$ *such that* $gtg^{-1} = t^{-1}$ *belongs to* $SO(Q)$*, i.e.,* $\det(g) = 1$*.*

Proof : We follow the notation in the previous lemma. Let $\dim(V) = 2m$, where m is even. As in the proof of the previous lemma, we may assume t is diagonal. Then any element T that conjugates t to t^{-1} over $\bar{k}$, is of the form

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

We have $\det(T) = (-1)^m \det(B)\det(C) = \det(B)\det(C) = 1$. Since g is a conjugate of T, the claim follows. □

Now we prove the main theorem about special orthogonal groups.

Theorem. *Let $t \in SO(Q)$ be a semisimple element. Then, t is real in $SO(Q)$ if and only if t is strongly real in $SO(Q)$.*

Proof : If $\dim(V) \not\equiv 2 \pmod 4$ then the every element of $SO(Q)$ is strongly real. Hence let us assume that $\dim(V) \equiv 2 \pmod 4$. Let $\dim(V) = 2m$ where m is odd. In this case we will prove that the element t is real in $SO(Q)$ if and only if 1 or -1 is an eigenvalue of t.

First we prove that if 1 and -1 are not eigenvalues then t is not real. It is enough to prove this statement over $\bar{k}$.

We write $\bar{V} = V \otimes_k \bar{k}$ and continue to denote t over $\bar{k}$ by t itself. We have a t-invariant orthogonal decomposition of $\bar{V}$;

$$\bar{V} = \bar{V}_1 \oplus \bar{V}_{-1} \oplus \bar{V}_{\lambda_1^{\pm 1}} \oplus \ldots \oplus \bar{V}_{\lambda_r^{\pm 1}}$$

where $\bar{V}_1$ and $\bar{V}_{-1}$ are the eigenspaces of t corresponding to 1 and -1 respectively and $\bar{V}_{\lambda_j^{\pm 1}} = \bar{V}_{\lambda_j} \oplus \bar{V}_{\lambda_j^{-1}}$ where $\bar{V}_{\lambda_j}$ is the eigenspace corresponding to λ_j for $\lambda_j^2 \neq 1$.

Since 1 and -1 are not eigenvalues for t, we have $\bar{V}_1 = 0$ and $\bar{V}_{-1} = 0$.

If $r = 1$ it follows from previous lemma that t is not real. Hence we may assume $r \geq 2$.

We denote the restriction of t on $\bar{V}_{\lambda_j^{\pm 1}}$ by t_j. Let the dimension of $\bar{V}_{\lambda_j^{\pm 1}}$ be n_j. Since $\lambda_j \neq \pm 1$, n_j is even and is either $0 \pmod 4$ or $2 \pmod 4$. Let the number of subspaces $\bar{V}_{\lambda_j^{\pm 1}}$ such that n_j is $2 \pmod 4$ be s. Then s is odd, since $\dim(V) \equiv 2 \pmod 4$. Let $g \in SO(Q)$ such that $gtg^{-1} = t^{-1}$. Then g leaves $\bar{V}_{\lambda_j^{\pm 1}}$ invariant for all j. We denote the restriction of g on $\bar{V}_{\lambda_j^{\pm 1}}$ by g_j.

Then $g_j \in O(\bar{V}_{\lambda_j^{\pm 1}})$ and $g_j t_j g_j^{-1} = t_j^{-1}$. From the previous lemma, determinant of g_j is 1 whenever $n_j \equiv 0 \pmod 4$ and the determinant of g_j is -1 whenever $n_j \equiv 2 \pmod 4$. Hence the determinant of g is $(-1)^s = -1$, which contradicts $g \in SO(Q)$. Hence t can not be real in $SO(Q)$.

Conversely, if 1 or -1 is an eigenvalue then the subspace $\bar{V}_1$ or $\bar{V}_{-1}$ is non-zero. These subspaces are defined over k. Let us denote their descents by V_1 and V_{-1} over k. The dimension of V_1 and V_{-1} is always even. But the matrix I and $-I$ can be written as a product of two involutions, each having determinant 1 or -1. Hence in this case t can be always written as a product of two involutions in $SO(Q)$. $\square$

Part-II

Reality in Linear Algebraic Groups

Some Notions in Algebraic Groups

- An algebraic group is a closed subgroup of GL_n for some n. The diagonal group D_n is a closed subgroup of GL_n which is isomorphic to $(GL_1)^n$.
- An algebraic group isomorphic to D_n is called an n-dimensional **torus**. Equivalently torus is a connected algebraic group consisting of commuting semisimple elements. Tori play important role in the study of structure of algebraic groups.
- A **maximal torus** T in an algebraic group G is a subgroup of G which is isomorphic to a torus and is not strictly contained in any other torus. All maximal tori in an algebraic group G are conjugate. And any semisimple element of G lies in a maximal torus.

• The radical of a group G is a maximal closed, connected, normal, solvable, subgroup denoted as $R(G)$ and the unipotent radical $R_u(G)$ is a maximal closed, connected, unipotent, subgroup of G. The group G is called **semisimple** if $R(G) = \{e\}$ and **reductive** if $R_u(G) = R(G)_u$ is trivial.

Example : The group GL_n is a reductive group which is not semisimple and SL_n is a semisimple group.

• For a maximal torus T in G the group $W = \frac{N_G(T)}{\mathcal{Z}_G(T)^o}$ is finite and is called **Weyl group** of G. For a reductive group G and a maximal torus T in G the centralizer $\mathcal{Z}_G(T) = T$ and we have following exact sequence:

$$\{1\} \longrightarrow T \longrightarrow N_G(T) \longrightarrow W = \frac{N_G(T)}{T} \longrightarrow \{1\}.$$

An element in G is called **regular** if its centralizer has minimal dimension among all centralizers. In a reductive group a semisimple element $x \in G$ is regular if and only if $\mathcal{Z}_G(x)^0$ is a maximal torus. An element is called **strongly regular** if its centralizer is a maximal torus.

Example 1 : In GL_n a semisimple element is regular if and only if it has all eigenvalues distinct.

Example 2 : Let $G = SO(3)$ and $t = \mathrm{diag}\{1, -1, -1\}$. Then $\mathcal{Z}_G(t) \cong O(2)$. The element t is regular but not strongly regular.

Reality Question for Linear Algebraic Groups

Let G be a connected, simple algebraic group defined over k (with $char(k) \neq 2$) such that the longest element w_0 in the Weyl group W of G, acts as -1. The groups of type A_1, B_l, C_l, $D_{2l} (l > 2), E_7, E_8, F_4, G_2$ are the groups which satisfy above hypothesis. We study reality of strongly regular elements over k in these groups. We also study reality in these groups over k using general cohomological methods and give proof for reality of semisimple elements of G over k with $cd(k) \leq 1$. Recall that a field k has $cd(k) \leq 1$, if and only if for every algebraic extension K of k, $Br(K) = 0$ ("Galois Cohomology" by Serre). Any C_1 field is an example of a field k with $cd(k) \leq 1$.

Cohomological Obstruction to Reality

Let G be a connected semisimple linear algebraic group. Let $t \in G$ be real. We denote $H = \mathcal{Z}_G(t)$ which is defined over k. Let us denote $X = \{x \in G \mid xtx^{-1} = t^{-1}\}$. Then X is an H-torsor defined over k with action $h.x = xh$ for $h \in H$. Note that t is real in $G(k)$ if and only if the H-torsor X has a k-point. We define a map from X to $H^1(k, H)$ by $x \mapsto \gamma_x$ where $\gamma_x \colon \Gamma \to H$ is defined by $\gamma_x(\sigma) = x^{-1}\sigma(x)$.

Lemma. *With notations as above, t is real in $G(k)$ if and only γ is a trivial cocycle in $H^1(k, H)$.*

Strongly Regular Elements and Reality

Let G be a connected simple, adjoint group defined over k. Assume that -1 belongs to the Weyl group of G.

From a theorem of Richardson and Springer (1990), any involution $c \in W(T)$ is represented by an involution n in $N(T)$.

As $-1 \in W$, we have $n \in N(T)$, an involution, which maps to -1 in the exact sequence

$$1 \to T \to N(T) \to W = N(T)/T \to 1.$$

Which means $nxn^{-1} = x^{-1}$ for all $x \in T$. And nx is an involution for all $x \in T$. Hence we get that in G (over $\bar{k}$) every semisimple element is real.

Theorem. *Let G be a group of the type mentioned above. Suppose $t \in G(k)$ is a strongly regular element in $G(k)$. Then, t is real in $G(k)$ if and only if t is strongly real in $G(k)$. Moreover, every element of a maximal torus, which contains a strongly real element, is strongly real.*

Proof: We have $t \in G(k)$ a strongly regular element. Let T be the maximal torus containing t defined over k. Then $\mathcal{Z}_G(t) = T$. Suppose t is real in $G(k)$, i.e., there exists $g \in G(k)$ such that $gtg^{-1} = t^{-1}$. Using Richardson and Springer we have $n \in G$ such that $nsn^{-1} = s^{-1}$. Then $g \in n\mathcal{Z}_G(t) = nT$, say $g = ns$. We check that $g^2 = nsns = s^{-1}s = 1$, i.e., g is an involution. Hence t is a product of two involutions. $\square$

Semisimple Elements over fields of $cd(k) \leq 1$

Now we consider groups of the type mentioned in the previous theorem. We take field k of $cd(k) \leq 1$. Then we have,

Theorem. *Let G be a simple adjoint group defined over k. Let w_0 be the longest element in the Weyl group acting as -1. Then every semisimple element in $G(k)$ is strongly real in $G(k)$.*

Proof : Let $t \in G(k)$ be a semisimple element. Let T be a torus in G defined over k which contains t, i.e., $t \in T(k)$. From a theorem of Richardson and Springer, as $-1 \in W$, there exists $n_0 \in N(T)$ with $n_0^2 = 1$ which represents -1 in W. That is, we have $n_0 s n_0^{-1} = s^{-1}$ for all $s \in T$. We claim that the coset $n_0 T$ is Γ-stable. We note that for $\sigma \in \Gamma = \text{Gal}(\bar{k}/k)$,

$$\sigma(n_0)s\sigma(n_0)^{-1} = \sigma(n_0\sigma^{-1}(s)n_0^{-1}) = \sigma(\sigma^{-1}(s^{-1})) = s^{-1}$$

for all $s \in T$ and $\sigma \in \Gamma$. Hence $\sigma(n_0) \in N(T)$ also represents -1 in W.

Thus we have $\sigma(n_0)T = n_0T$ and so $n_0\sigma(n_0) \in T$.

We look at the cocycle defined by $\sigma \mapsto n_0\sigma(n_0)$. Then the image of this cocycle lands in T. Since $cd(k) \leq 1$, from a theorem of Steinberg, we have $H^1(k,T) = 0$ and hence the cocycle defined above is a trivial cocycle. That is, there exists $t_0 \in T$ such that $n_0\sigma(n_0) = t_0\sigma(t_0^{-1})$ for all $\sigma \in \Gamma$. This implies $\sigma(n_0t_0) = n_0t_0$ for all $\sigma \in \Gamma$ and hence $n_0t_0 \in G(k)$. We check that n_0t_0 is an involution and conjugates every element of T to its inverse.

$$(n_0t_0)^2 = n_0t_0n_0t_0 = t_0^{-1}t_0 = 1$$

and

$$n_0t_0s(n_0t_0)^{-1} = n_0t_0st_0^{-1}n_0 = n_0sn_0 = s^{-1}.$$

Hence every semisimple element of $G(k)$ is real in $G(k)$. $\square$

Part-III

Groups of Type G_2

Groups of Type G_2

A group G of type G_2 over a field k can be realized as a group of k-automorphisms of an octonion algebra over k. To define octonion algebras we need notion of a **composition algebra** over a field k.

Definition (Composition Algebra). *A composition algebra* $\mathfrak{C}$ *over a field* k *is an algebra over* k*, not necessarily associative, with an identity element* 1 *together with a non-degenerate quadratic form* N *on* $\mathfrak{C}$*, permitting composition, i.e.,*

$$N(xy) = N(x)N(y) \;\; \forall\, x, y \in \mathfrak{C}.$$

The quadratic form N is called the **norm** on $\mathfrak{C}$. The possible dimensions of a composition algebra are $1, 2, 4, 8$. The algebras of dimension 8 are neither commutative nor associative (called **octonion** algebras or **Cayley** algebras).

Let $\mathfrak{C}$ be an octonion algebra.

Proposition. *The algebraic group* $\mathcal{G} = \mathrm{Aut}(\mathfrak{C}_K)$*, where* $\mathfrak{C}_K = \mathfrak{C} \otimes K$ *and* K *is an algebraic closure of* k*, is the split, connected, simple algebraic group of type* G_2*. Moreover, the automorphism group* $\mathcal{G}$ *is defined over* k*.*

In fact (see "Galois Cohomology" by Serre), any simple group of type G_2 over a field k is isomorphic to the automorphism group of an octonion algebra $\mathfrak{C}$ over k.

G_2 is Either Anisotropic or Split

There is a dichotomy with respect to the norm of octonion algebras (in general, for composition algebras). The norm N is a **Pfister** form (tensor product of norm forms of quadratic extensions) and hence is either anisotropic or hyperbolic. If N is anisotropic, every nonzero element of $\mathfrak{C}$ has an inverse in $\mathfrak{C}$ and is a **division** octonion algebra. We call the corresponding group of type G_2 an anisotropic group. If N is hyperbolic, up to isomorphism, there is only one octonion algebra with N as its norm, called the **split** octonion algebra and the group is called split group.

Real Elements in G_2

Theorem. *In addition, if char*$(k) \neq 3$, *every unipotent element in* $G(k)$ *is strongly real in* $G(k)$.

For a general element in $G(k)$, we prove,

Theorem. *Let char*$(k) \neq 2, 3$. *Then, an element* g *is real in* $G(k)$ *if and only if it is strongly real in* $G(k)$.

Cayley-Dickson Doubling

Let $\mathfrak{C}$ be a composition algebra and $\mathfrak{D} \subset \mathfrak{C}$ a composition subalgebra and $\mathfrak{D} \neq \mathfrak{C}$. Let $a \in \mathfrak{D}^{\perp}$ with $N(a) = -\lambda \neq 0$. Then

$$\mathfrak{D}_1 = \mathfrak{D} \oplus \mathfrak{D}a$$

is a composition subalgebra of $\mathfrak{C}$ of dimension $2dim(\mathfrak{D})$. The product on $\mathfrak{D}_1$ is given by

$$(x + ya)(u + va) = (xu + \lambda\overline{v}y) + (vx + y\overline{u})a,$$

$x, y, u, v \in \mathfrak{D}$ where $x \mapsto \overline{x}$ is the involution on $\mathfrak{D}$. The norm on $\mathfrak{D}_1$ is given by

$$N(x + ya) = N(x) - \lambda N(y).$$

Some Subgroups of the Group G_2

Let $\mathfrak{C}$ be an octonion algebra over a field k of characteristic $\neq 2$. Let L be a composition subalgebra of $\mathfrak{C}$. We define

$$G(\mathfrak{C}/L) = \{t \in \mathrm{Aut}(\mathfrak{C}) \mid t(x) = x \;\forall\; x \in L\}$$

and

$$G(\mathfrak{C}, L) = \{t \in \mathrm{Aut}(\mathfrak{C}) \mid t(x) \in L \;\forall\; x \in L\}\,.$$

Jacobson studied $G(\mathfrak{C}/L)$ in his paper titled "Composition Algebras and their Automorphisms"(1958).

Let L be a two dimensional composition subalgebra of $\mathfrak{C}$. Then L is either a quadratic field extension of k or $L \cong k \times k$.

Subgroups of Type $SU(V, h)$

Let us assume first that L is a quadratic field extension of k and $L = k(\gamma)$, where $\gamma^2 = c.1 \neq 0$. Then $L^\perp$ is a left L vector space via the octonion multiplication. Also,

$$h\colon L^\perp \times L^\perp \longrightarrow L$$

$$h(x, y) = N(x, y) + \gamma^{-1} N(\gamma x, y),$$

is a non-degenerate hermitian form on $L^\perp$ over L.

Proposition (Jacobson). *In this case, the subgroup $G(\mathfrak{C}/L)$ of G is isomorphic to the unimodular unitary group $SU(L^\perp, h)$ of the three dimensional space $L^\perp$ over L relative to the hermitian form h, via the isomorphism,*

$$\begin{aligned} \psi\colon G(\mathfrak{C}/L) &\longrightarrow SU(L^\perp, h) \\ t &\longmapsto t|_{L^\perp}. \end{aligned}$$

Subgroups of Type SL_3

Now, let us assume that L is a split two dimensional étale subalgebra of $\mathfrak{C}$. Then $\mathfrak{C}$ is necessarily split and L contains a nontrivial idempotent e. There exists a basis $B = \{1, u_1, u_2, u_3, e, w_1, w_2, w_3\}$ of $\mathfrak{C}$, called the **Peirce basis** with respect to e, such that the subspaces $U = \text{span}\{u_1, u_2, u_3\}$ and $W = \text{span}\{w_1, w_2, w_3\}$ satisfy $U = \{x \in \mathfrak{C} \mid ex = 0,\ xe = x\}$ and $W = \{x \in \mathfrak{C} \mid xe = 0,\ ex = x\}$. We have, for $\eta \in G(\mathfrak{C}/L)$ we have $\eta(U) = U$ and $\eta(W) = W$. Then we have,

Proposition. *In this case $G(\mathfrak{C}/L)$ is isomorphic to the unimodular linear group $SL(U)$, via the isomorphism given by,*

$$\begin{aligned} \phi\colon G(\mathfrak{C}/L) &\longrightarrow SL(U) \\ \eta &\longmapsto \eta|_U . \end{aligned}$$

Subgroups of Type SL_2

Let $\mathfrak{D} \subset \mathfrak{C}$ be a quaternion subalgebra. Then we have, by Cayley-Dickson doubling,

$$\mathfrak{C} = \mathfrak{D} \oplus \mathfrak{D}a$$

for some $a \in \mathfrak{D}^{\perp}$ with $N(a) \neq 0$. Let $\phi \in \mathrm{Aut}(\mathfrak{C}, \mathfrak{D})$. Then for $z = x + ya \in \mathfrak{C}$, there exists $c, p \in \mathfrak{D}$ with $N(c) \neq 0$ and $N(p) = 1$ such that

$$\phi(z) = cxc^{-1} + (pcyc^{-1})a.$$

Hence, we have,

Proposition. *The group of automorphisms of $\mathfrak{C}$, leaving $\mathfrak{D}$ point-wise fixed, is isomorphic to $SL_1(\mathfrak{D})$, the group of norm 1 elements of $\mathfrak{D}$. In the above notation, $G(\mathfrak{C}/\mathfrak{D}) \cong SL_1(\mathfrak{D})$.*

Sketch of the Proof of Main Theorem

Let G be a group of type G_2 defined over a field k of characteristic $\neq 2$. Then, there exists an octonion algebra $\mathfrak{C}$ over k such that $G \cong \mathrm{Aut}(\mathfrak{C})$. Let t_0 be a semisimple element of $G(k)$. We will also denote the image of t_0 in $\mathrm{Aut}(\mathfrak{C})$ by t_0. We write $\mathfrak{C}_0$ for the subspace of trace 0 elements of $\mathfrak{C}$. We put

$$V_{t_0} = ker(t_0 - 1)^8.$$

Then V_{t_0} is a composition subalgebra of $\mathfrak{C}$ with norm as the restriction of the norm on $\mathfrak{C}$ (due to Wonenburger). Let $r_{t_0} = dim(V_{t_0} \cap \mathfrak{C}_0)$. Then r_{t_0} is $1, 3$ or 7.

We note that if $r_{t_0} = 7$, the characteristic polynomial of t_0 is $(X-1)^8$ and t_0 is unipotent. We have,

Lemma. *Let $t_0 \in G(k)$ be a unipotent element. In addition, we assume $char(k) \neq 3$. Then t_0 is strongly real in $G(k)$.*

Lemma. *Let the notation be as fixed above and let $t_0 \in G(k)$ be an element which is not unipotent (e.g. a semisimple element). Then, either t_0 leaves a quaternion subalgebra invariant or fixes a quadratic étale subalgebra L of $\mathfrak{C}$ pointwise. In the latter case, $t_0 \in SU(V,h) \subset G(k)$ for a rank 3 hermitian space V over a quadratic field extension L of k or $t_0 \in SL(3) \subset G(k)$.*

Wonenburger in her paper "Automorphism of Cayley Algebras"(1969) proved that , if t_0, an automorphism of $\mathfrak{C}$, leaves a quaternion subalgebra invariant, it is a product of two involutions and hence real in $G(k)$.

We discuss the other cases now, i.e., t_0 leaves a quadratic étale subalgebra L of $\mathfrak{C}$ point-wise fixed.

1. The fixed subalgebra L is a quadratic field extension of k and
2. The fixed subalgebra is split, i.e., $L \cong k \times k$.

Let us denote the image of t_0 by A in $SU(L^\perp, h)$ or in $SL(3)$ as the case may be. Let us denote by $\chi_A(X)$, the characteristic polynomial and by $m_A(X)$ the minimal polynomial of A over L in the first case and over k in the second.

case 1: If $\chi_A(X) \neq m_A(X)$ then t_0 leaves a quaternion algebra invariant and hence strongly real.

case 2: Let $\chi_A(X) = m_A(X)$ then we prove in the first case t_0 is conjugate to t_0^{-1} if and only if $\bar{A}$ is conjugate to A^{-1} in $SU(3)$ and in the second case A is conjugate to ${}^tA$ in SL_3.

Combining these results with that of Neumann and Wonenburger we get the required results.

Part-IV

Conclusion and Questions

Reality in Algebraic Groups

- There are still a lot to be answered in classical groups.
- One also needs to study other exceptional groups other than G_2.
- The results obtained here explicitly for several groups indicate better general results for semisimple algebraic groups than proved here.
- One expects a statement for semisimple elements similar to strongly regular elements.

Representation Theory and Real Elements

Proposition. *Let G be a finite group. The number of real irreducible characters of G is equal to the number of real conjugacy classes of G.*

Question: Is there any relation between orthogonal representations and strongly real conjugacy classes?

Comparing with the results of D. Prasad (1998 and 1999 on self-dual representations) about determining groups of Lie type and p-adic groups for which all self-dual representations are orthogonal, one expects it to be related to all real elements being strongly real.

Thank You.