third chapter, besides the standard topics on topological vector spaces (Hahn-Banach, Krein-Milman, duality), there is a special emphasis on topics pertaining to completeness (hence to uniformity). The fourth chapter includes standard material on normed algebras and Banach algebras, with and without involution, as well as not-so-standard material on more esoteric topological algebras (locally m-convex Q-algebras, etc.). The fifth and final chapter culminates in a proof of the Pontrjagin duality theorem. The prerequisites for reading the book (drawn mainly from general topology and integration theory) are sketched in three brief appendices.

Proofs are detailed and carefully done. The layout is excellent; the printer deserves a medal for his skill in representing the many, often intricate and unusual, notations. The text is heavy on special symbols and terminology; since these are usually defined once and used from then on without explanation, the burden on the reader's power of concentration builds quickly. (The burden on the proofreader's concentration was more than occasionally overwhelming.) An index of symbols and a good general index help, but the reader's task is still formidable (the browser's, hopeless).

The author states in his Preface: "This work is reasonably self-contained and accessible to students with a background in elementary analysis, linear algebra and point set topology. At the same time it covers a good amount of advanced material without going off into the purple deep." The reviewer concurs; there is a lot of fine material in this book for second-year graduate courses and seminars.

Every mathematician needs to speak a little topology. The message of this book is that every analyst needs to speak a little uniformity; it is a central language of analysis, not just a peripheral dialect. Uniform structures deserve a niche in every first-year graduate course in general topology; this book effectively demonstrates why.

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General theory of Lie algebras, by Yutze Chow, Gordon and Breach, New York, 1978, Volume 1, xxii + 461 pp., Volume 2, xx + 436 pp., $72.00.

1. Among the three main types of nonassociative algebras, Lie, alternative and Jordan algebras, the Lie algebras were the first to be studied and are still the most important because of their connections with other parts of mathematics. In fact the Lie algebras were devised to study the Lie groups and appeared for the first time in the work of the Norwegian mathematician Sophus Lie (1842–1899).

In order to analyze the structure of a continuous group a linear structure called an infinitesimal group is deduced when a neighborhood of the identity element of the group is known. The advantage is that infinitesimal groups can be studied using linear algebra and that the local properties of the group are reflected in the structure of its infinitesimal group. So, until the nineteen thirties when Herman Weyl introduced the term Lie algebras, these algebras lived under the name of infinitesimal groups as part of the study of continuous groups. The new name gave a broader and independent life to the Lie algebras that started to be studied on their own without considering their relation to the Lie groups. This is the approach taken in Professor Chow's book.

A vector space L over a field F with a binary operation or product linear in each variable is called a linear algebra. Let us denote the product of the elements x and y of L by $[xy]$. The linear algebra L is called a Lie algebra if the product satisfies the following two axioms.

(1) $[xx] = 0$ for all $x \in L$. Because this condition together with linearity and characteristic different from 2 is equivalent to $[xy] = -[yx]$ it is called anticommutativity.

Instead of the associative law the product satisfies the following property called the Jacobi identity.

(2) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all x, y and z in L.

The use of brackets [ ] to denote the product in a Lie algebra is a general practice which is not always followed in the book under review.

When F is the field of real or complex numbers, we talked about real or complex Lie algebras, respectively. The infinitesimal groups were always real or complex Lie algebras.

The most familiar example of a real Lie algebra is the cross product in 3-dimensional real space. This is the Lie algebra or infinitesimal group of the real Lie group $SO(3, \mathbf{R})$, the group of rotations in a 3-dimensional Euclidean space.

To become friendly with Lie algebras let us get some recipes for concocting such algebras and give examples of some well-known members of the family.

EXAMPLE 1. Let L be any vector space. Define $[xy] = 0$ for all x and y in L. With this bracket operation L becomes a Lie algebra. Such Lie algebras are called abelian Lie algebras. This is a good name because the real and complex abelian Lie algebras are the infinitesimal groups of the abelian Lie groups.

Now here comes a recipe. Take any associative algebra and let ab stand for the product of the elements a and b of A. To say that A is associative means that it is a linear algebra with $(ab)c = a(bc)$ for all a, b, c in A. Define the Lie product [ ] as follows, $[ab] = ab - ba$. With this bracket operation the algebra A becomes the Lie algebra A_L.

EXAMPLE 2. Let us take the associative algebra of $n \times n$ matrices with coefficients in the field F. The Lie algebra obtained from it by introducing the bracket operation is called the general linear algebra over F, $gl(n, F)$. The real and complex general linear algebras $gl(n, \mathbf{R})$ and $gl(n, \mathbf{C})$ are the Lie algebras of the real and complex Lie groups of invertible $n \times n$ matrices, respectively.

Any vector subspace of a Lie algebra closed under the bracket operation is again a Lie algebra.

EXAMPLE 3. The vector space of $n \times n$ matrices with real or complex coefficients whose trace is zero is closed under the bracket operation. This subalgebra of the general linear algebra is called the special linear algebra and denoted $\mathrm{sl}(n, \mathbf{R})$, if the matrices are real, and $\mathrm{sl}(n, \mathbf{C})$, if we are dealing with complex matrices. They are the Lie algebras of the special linear groups of $n \times n$ matrices of determinant 1 with real and complex coefficients, respectively.

Let us get a recipe to define a subspace of an associative algebra A with involution which is closed under the Lie product. Suppose that A has an antiautomorphism τ of period 2. That is, τ is a linear mapping of A into itself such that $\tau(ab) = \tau(b)\tau(a)$ and τ^2 is the identity, such τ is also called an involution. Let S be the subspace of A consisting of τ-skew elements, i.e., $S = \{a \in A | \tau(a) = -a\}$. It is immediate to see that, if a and b are in S, $[ab]$ is in S. Hence S is a Lie algebra under the Lie product.

EXAMPLE 4. Let M be an $n \times n$ matrix and $\tau(M) = M^t$, the transpose of M. Then τ is an involution of the algebra of $n \times n$ matrices with coefficients in F and the τ-skew elements are the skew-symmetric matrices which, therefore, form a Lie algebra under the bracket operation. When F is the complex field we get the complex orthogonal algebra $\mathfrak{o}(n, \mathbf{C})$. It corresponds to the complex orthogonal Lie group $\mathrm{O}(n, \mathbf{C})$ consisting of complex matrices T satisfying $T^t = T^{-1}$.

EXAMPLE 5. Let us define a different involution in the algebra of $2n \times 2n$ matrices. Take the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where 0 and I_n stand for the $n \times n$ zero and unit matrices, respectively. Notice that $BB^t = -I_{2n}$. For any $2n \times 2n$ matrix M define $\tau(M) = BM^tB^{-1}$. Then τ is an involution and the Lie algebra of τ-skew matrices is called the symplectic algebra $\mathrm{sp}(2n, F)$. When F is $\mathbf{R}$ or $\mathbf{C}$ we get the real and complex symplectic groups $\mathrm{Sp}(2n, \mathbf{R})$ and $\mathrm{Sp}(2n, \mathbf{C})$. These groups consist of all real or complex matrices T such that $TBT^t = B$.

To a given structure it corresponds a group of automorphisms or symmetries. To a linear algebra A we can also attach a Lie algebra, called the algebra of derivations, that we are going to define. Let ab stand for the product of the elements a and b of A, a linear mapping D of A into itself is called a derivation if $D(ab) = D(a)b + aD(b)$. A straightforward computation shows that, if D_1 and D_2 are derivations of A, then any linear combination of these two linear transformations is a derivation, as well as their Lie product $[D_1D_2] = D_1D_2 - D_2D_1$. Hence the space $\mathrm{Der}(A)$ of all derivations of A is a Lie algebra.

Two examples of exceptional Lie algebras are obtained as algebras of derivations of two eccentric members of the families of alternative and Jordan algebras.

If A is an associative algebra the function $f(x, y, z)$ defined on $A \times A \times A$ by $f(x, y, z) = (xy)z - x(yz)$ is always zero. Those linear algebras A for which $f(x, y, z)$ is an alternating function are called alternative algebras, so the alternative algebras include the associative algebras. They originated in the observation made by E. Artin that an eight dimensional real linear algebra defined independently by J. T. Graves and A. Caley in the eighteen forties, although not associative, satisfies the condition mentioned above. The study of alternative algebras showed that it is the only nonassociative alternative real division algebra, i.e., it has an identity and every nonzero element has a two-sided inverse. Another very important property is that it has an involution J such that for any element x the product xx^J is a scalar multiple of the identity. We will call this algebra the Cayley numbers and will denote it by $\mathfrak{O}$ because it is also known as the algebra of octaves or octonions. The history of this algebra and its connections with other parts of mathematics are discussed in F. van der Blij article *History of the octaves*, Simon Stevin (1961).

The Jordan algebras are named after the physicist P. Jordan and their birth certificate is the joint paper by P. Jordan, J. von Neumann and E. Wigner entitled *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math. (1934).

The ordinary product of two hermitian matrices or two hermitian operators A and B is not hermitian; to make it so one can try the "Jordan" product $A \times B = \frac{1}{2}(AB + BA)$. With this product we obtain a commutative (nonassociative) multiplication such that $(A \times B) \times A^2 = A \times (B \times A^2)$.

Any commutative linear algebra whose product satisfies this equality is called a Jordan algebra. Not every Jordan algebra can be obtained from an associative algebra by introducing the "Jordan" product. We are going to consider a very special example.

Take the algebra $M_3(\mathfrak{O})$ of 3×3 matrices with coefficients in $\mathfrak{O}$, the Cayley members. If A is a matrix in $M_3(\mathfrak{O})$, let A^* be its J-conjugate transpose. The set $\mathcal{H}(\mathfrak{O}) = \{A \in M_3(\mathfrak{O}) | A = A^*\}$ of hermitian matrices of $M_3(\mathfrak{O})$ under the multiplication defined by $A \times B = \frac{1}{2}(AB + BA)$ is a Jordan algebra. Because the algebra $\mathfrak{O}$ is not associative, the hermitian matrices of $M_n(\mathfrak{O})$ do not form a Jordan algebra under $\times$ when $n > 3$.

We will say something below about the derivation algebras of $\mathfrak{O}$ and $\mathcal{H}(3, \mathfrak{O})$.

2. In spite of the great growth and development of the theory of Lie algebras in the present century, the most surprising and central results in this theory are straight generalizations of the results contained in a series of papers (1888–1890) by W. Killing, and the justly famous thesis of E. Cartan of 1894: the classification of the finite dimensional complex simple Lie algebras and their finite dimensional irreducible representations.

From now on Lie algebra will mean a finite dimensional Lie algebra over a field of characteristic zero.

If x is an element of a Lie algebra L the mapping of L into itself that takes y into $[xy]$ is a linear transformation called the adjoint of x and denoted ad x. With this notation we get $[xy] = \operatorname{ad} x(y)$ and $\operatorname{ad}[xy] = [\operatorname{ad} x, \operatorname{ad} y]$. Now we define a symmetric bilinear form K on L called the Killing form by taking

$K(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \cdot \mathrm{ad}\, y)$, the trace of the linear transformation $\mathrm{ad}\, x \cdot \mathrm{ad}\, y$. It follows from the definition that the Killing form is "associative", this means that $K([xy], z) = K(x, [yz])$. We say that L is semisimple if its Killing form is nondegenerate. From the associativity of K it is deduced that any semisimple L can be decomposed uniquely into a direct sum $L = L_1 \oplus L_2 \oplus \cdots \oplus L_r$, where $[x_i y_j] = 0$ for all x_i in L_i and y_j in L_j, if $i \neq j$, and each L_i is indecomposable. Moreover each L_i is also semisimple.

Such indecomposable semisimple algebras are called simple. If G is a complex simple Lie group its Lie algebra is simple. The aim of the classification of the complex simple Lie algebras was to find the different classes of complex simple Lie groups.

To get a closer look into a simple Lie algebra we are going to break it into a direct sum of subspaces that reveals part of its structure as an algebra. But to use the full power of the methods of linear algebra we must *assume now that the field F is algebraically closed* or may consider, if interested only in the complex simple Lie algebras, that we are dealing with the field of complex numbers. Then any linear transformation whose minimal polynomial has no multiple roots is diagonalizable, and for any set S of commuting diagonalizable linear transformation acting on a finite dimensional space there is a basis consisting of eigenvectors.

Let H be a subalgebra of L maximal with respect to the property that for every $h \in H$ ad h is diagonalizable. We call such subalgebra a Cartan subalgebra. It turns out that H is abelian, the restriction of the Killing form to H is nondegenerate, and $L = H \oplus \sum L_\alpha$, where $\sum L_\alpha$ is a direct sum of 1 dimensional eigenspaces L_α. That is, if $x_\alpha \in L_\alpha$, $[hx_\alpha] = \mathrm{ad}\, h(x_\alpha) = \alpha(h)x_\alpha$, where $\alpha(h)$ is the eigenvalue of ad h. Thus α is an element of the dual H^* of H, for $\alpha(h)$ depends linearly on h.

The nonzero elements α of H^* such that $L_\alpha = \{x \in L | \mathrm{ad}\, h(x) = \alpha(h)x$ for all $h \in H\} \neq 0$ are called roots; if α is a root, $\dim L_\alpha = 1$. The set of roots spans H^* and can be represented as a spanning set Φ of vectors in an l dimensional Euclidean space E, where $l = \dim H$. This is done in the manner illustrated in the example given below.

The set of vectors Φ satisfies the following conditions:

(1) If $\alpha \in \Phi$, then $-\alpha \in \Phi$, but $m\alpha \notin \Phi$, if $m \neq \pm 1$.

(2) The reflections in any of the hyperplanes P_α, $\alpha \in \Phi$, going through the origin and orthogonal to α take Φ into Φ.

(3) If (x, y) stands for the inner product of the vectors x and y in E, for any two vectors α and β in Φ, $2(\alpha, \beta)/(\beta, \beta)$ is an integer called a Cartan integer.

(4) Φ is irreducible in the sense that it cannot be partitioned into two sets Φ_1 and Φ_2 so that every vector in Φ_1 is orthogonal to every vector in Φ_2.

A spanning set of vectors in a Euclidean space satisfying conditions 1–4 is called an irreducible root system. The classification of the simple Lie algebras over an algebraically closed field of characteristic zero boils down to finding all possible irreducible root systems. It turns out that there are four infinite families of irreducible root systems and five exceptional ones. There is a simple Lie algebra corresponding to each irreducible root system.

The group of Euclidean transformations generated by the reflections in the hyperplanes P_α defined by a root system Φ, is called the Weyl group of Φ and is always finite. Some of these groups are the groups of symmetries of regular polytopes. In such cases the hyperplanes P_α are the complete set of hyperplanes of symmetry of the regular polytope.

EXAMPLE. Let us cut open the complex simple Lie algebra sl(3, **C**) of 3×3 matrices of trace zero, to bring forth its root system.

Its two dimensional subspace consisting of the diagonal matrices of trace zero, is a Cartan subalgebra H. Relative to H, the space

$$\mathrm{sl}(3, \mathbf{C}) = H \oplus \mathbf{C}E_{12} \oplus \mathbf{C}E_{23} \oplus \mathbf{C}E_{13} \oplus \mathbf{C}E_{21} \oplus \mathbf{C}E_{32} \oplus \mathbf{C}E_{31},$$

where E_{ij} stands for the matrix with 1 in the (i, j) position and zero elsewhere, and the $\mathbf{C}E_{ij}$'s are the one dimensional eigenspaces. For, if the diagonal matrix $D = \mathrm{diag}(a_1, a_2, a_3)$, with $a_1 + a_2 + a_3 = 0$, represents the general element of H, then

$$\mathrm{ad}\, D(E_{ij}) = [DE_{ij}] = (a_i - a_j)E_{ij},$$

and the linear functional α_{ij} corresponding to $\mathbf{C}E_{ij}$ is defined by $\alpha_{ij}(D) = a_i - a_j$. Hence $\alpha_{21} = -\alpha_{12}$, $\alpha_{32} = -\alpha_{23}$, $\alpha_{31} = -\alpha_{13}$, and, $\alpha_{13} = \alpha_{12} + \alpha_{23}$, but α_{12} and α_{23} are linearly independent.

If $D' = \mathrm{diag}(a_1', a_2', a_3')$, with $a_3' = -(a_1' + a_2')$, we find that the value of the Killing form

$$\begin{aligned} K(D, D') &= 2(a_1 - a_2)(a_1' - a_2') + 2(a_1 - a_3)(a_1' - a_3') + 2(a_2 - a_3)(a_2' - a_3') \\ &= 2(a_1 - a_2)(a_1' - a_2') + 2(2a_1 + a_2)(2a_1' + a_2') + 2(a_1 + 2a_2)(a_1' + 2a_2') \\ &= 6\,\mathrm{Tr}(DD'), \end{aligned}$$

which shows that K is positive definite when restricted to real diagonal matrices. Since K restricted to H is nondegenerate, it defines a canonical identification of H and its dual H^*. Denoting by $(\cdot, \cdot)$ the restriction of K to the real plane $\mathbf{R}\alpha_{12} + \mathbf{R}\alpha_{23} \subset H^* = H$, this plane becomes a Euclidean plane. For, under the canonical identification of H with H^*, $\alpha_{12} = \frac{1}{6}\,\mathrm{diag}(1, -1, 0)$ and $\alpha_{23} = \frac{1}{6}\,\mathrm{diag}(0, 1, -1)$. Then the Cartan numbers $2(\alpha_{12}, \alpha_{23})/(\alpha_{23}, \alpha_{23})$ and $2(\alpha_{23}, \alpha_{12})/(\alpha_{12}, \alpha_{12})$ are both equal to -1, and the six end points of the six vectors $\{\pm\alpha_{12}, \pm\alpha_{23}, \pm\alpha_{13}\}$ of the root system are the vertices of a regular hexagon. The three lines (hyperplanes) through the origin perpendicular to these vectors are the axes of symmetries of an equilateral triangle, so that the Weyl group of the root system is the dihedral group D_3.

A similar dissection of the complex simple Lie algebra sl(n, **C**), $n > 1$, gives a root system whose Weyl group is the group of symmetries of an $n - 1$ dimensional regular simplex. The group is isomorphic to the symmetric group S_n.

The complex Lie algebras whose root systems belong to one of the four infinite families are identified as the special linear, orthogonal and symplectic Lie algebras of Examples 3, 4 and 5 of §1. The root systems of the orthogonal Lie algebras split into two families, because the orthogonal groups of even dimensional spaces behave differently than the ones corresponding to spaces of odd dimension. As for the five exceptional root systems, their corresponding Lie algebras can be constructed from the information obtained in the analysis of the structure of the simple Lie algebras (in the book we are considering the constructions carried out are due to H. Freudental).

Two of the exceptional complex Lie algebras have been identified as the complexications (allowing complex coefficients) of the derivation algebras of the Cayley numbers and the Jordan algebra $\mathcal{H}(3, \mathfrak{O})$ mentioned in §1. Using sophisticated constructions based on the Cayley numbers and certain Jordan algebras, it is possible to obtain a unified description of the five exceptional complex simple Lie algebras.

The existence of the exceptional algebras was completely unsuspected before the classification of the complex simple Lie algebras. Their analogues over arbitrary fields are responsible for the intrusion of Lie theory into other parts of algebra, especially the description of the known families of non-abelian finite simple groups.

Returning now to our complex simple Lie algebras, we are faced with a problem. Does the type of root system that we obtain for a Lie algebra L depend on the Cartan subalgebra that has been picked up? The answer is no, because it can be proved than any two Cartan subalgebras are conjugate under an automorphism of L, which implies that the two root systems are isomorphic.

What about the finite dimensional representations of the simple complex Lie algebras? In the 1920's H. Weyl using the connection between Lie algebras and compact groups, proved that any finite-dimensional representation of a complex simple Lie algebra is completely reducible. The result is known as Weyl's theorem and his method as the unitary trick. The first purely algebraic proof of the theorem was given by Casimir and Van der Waerden in the 1930's, and is valid for simple Lie algebras over any field of characteristic zero. The theorem implies that it suffices to find the irreducible representations of the simple Lie algebras of characteristic zero. But E. Cartan had already proved that there is a bijection between the irreducible representations of a complex simple Lie algebra and a special type of linear forms defined on a Cartan subalgebra and called dominant weights.

3. The general definitions and basic theorems of the theory of Lie algebras, together with the construction and study of the universal enveloping algebra, the Levi decomposition of a Lie algebra and Weyl's theorem on complete reducibility form the first volume of the book under review.

The results leading to and including the classification and construction of the complex simple Lie algebras and their irreducible finite-dimensional representations, plus a long chapter on some cohomological and functorial properties of Lie algebras containing most of the basic definitions of category theory, constitute the second volume of the book.

Several books and lecture notes covering this material and much more are mentioned in the bibliography which appears in the second volume. Professor Chow's book differs from them among other things by some peculiarities in the notation and terminology, the rejection of a universally used simplified notation for modules and representations, the repetition of proofs using distinct notations and the amount of computational detail included in the proofs, which account in part for the length of the book.

The book contains no exercises, partly because the routine and straightforward computations that are usually left to the reader are worked out in the text, and partly because it is not the intention of the author to refer to results in the theory of Lie algebras not covered in the text.

Unfortunately, there is a large number of misprints in the text and many symbols have been left out. Apart from some wrong definitions, the table containing the Coxeter-Dynkin diagrams for the complex simple algebras gives the wrong diagram for E_8. In the bibliography the name of J. Dixmier is misspelled and some French words have wrong accents.

The efforts of the author to bring down an important subject to the level of readers without a general background in Mathematics are commendable, but any reader of this book may get lost in its forest of mathematical equations.

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Homology and cohomology theory, by William S. Massey, Monographs and Textbooks in Pure and Applied Mathematics, no. 46, Marcel Dekker, Inc., New York, 1978, xiv + 412 pp. $29.75.

Algebraic topology attempts to solve topological problems using algebra. To do so requires some sort of machine which produces the "algebraic image" of topology, and it is the machine itself on which topologists often spend most of their time, first carefully building and then diligently refining. Historically, the first such machine was ordinary homology and cohomology theory. (The word "ordinary" is not a slur–it means homology and cohomology defined from an algebraic chain complex as opposed to "extraordinary" theories such as K-theory.) Yet in the 84 years since homology was first mentioned, algebraic topology has developed rapidly and diversely. In the past 30 years this development has been particularly apparent with the problems more diverse and the machinery more and more complex, going far beyond its humble origins. Indeed, at the moment the subject is a tinkerer's delight; one can choose a machine and modify it almost at will.

Precisely when the machine works; or how it is related to other parts of the subject, is often not quite known. The phrase "nice space" seems to be used with increasing frequency in algebraic topology. What is the most efficient way to develop the machinery? What is the best way to teach it to graduate students or to explain it to other mathematicians? All this is often forgotten in the frenzy to answer the next question. Exposition and careful development of the foundations have often appeared in unpublished lecture notes, and copies turn into prized possessions. Sad to say, there is no glory in cleaning up after a party.

Homology and cohomology theory is a cleansing performed at the very roots of algebraic topology. It develops ordinary homology and cohomology theory in a neat and orderly fashion from the beginning, and does so in a novel way which is technically very pleasant. But why should such an old and established area of topology require cleansing at all? The answer lies in the