Simultaneous Diagonalization of Symmetric Bilinear Forms

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In the present note we study conditions under which two non-degenerate symmetric bilinear forms can be simultaneously diagonalized (see Theorem 1 below). Then we consider the possibility of having degenerate symmetric bilinear forms and apply our findings to the case in which the base field is real-closed and the forms do not vanish simultaneously. We obtain in this way a purely algebraic proof of a result of Milnor (Theorem 2). For non-algebraic proofs of this theorem the reader is referred to [2, 158–163] and [1]. These proofs assume that the base field is the real field and the validity of the results for any real-closed field follows from Tarski principle. The algebraic proof avoids the use of this principle.

1. Let M be a finite dimensional vector space with two non-degenerate symmetric bilinear forms $(x, y)_1$ and $(x, y)_2$. Assume that both forms can be simultaneously diagonalized, that is, there exists a basis of M, $x_1, x_2, \cdots, x_n$, such that the matrix of the form $(x, y)_i$ with respect to this basis has diagonal form for $i = 1, 2$. If for any subset S of M we denote by $[S]$ the subspace spanned by the element of S, we can write

$$M = [x_1] \oplus [x_2] \oplus \cdots \oplus [x_n],$$

and this is a decomposition of M in orthogonal subspaces.

Let M^* be the dual of M and let $\langle y^*, x\rangle$ be the value of $y^* \in M^*$ at x. Now, if φ_i is the bijective mapping of M into its dual defined by $\langle y\varphi_i, x\rangle = (y, x)_i$, we get $x_i\varphi_1\varphi_2^{-1} = \alpha_i x_i$, if x_i is one of the elements of the basis given above. This says that the vectors x_i, $i = 1, 2, \cdots, n$, are characteristic vectors of the linear automorphism of $M\varphi_1\varphi_2^{-1}$.

Theorem 1. *Let M be a finite dimensional vector space with two non-degenerate bilinear forms $(x, y)_1$ and $(x, y)_2$. Then, if the base field has characteristic $\neq 2$, the two forms can be simultaneously diagonalized if and only if M has a basis consisting of characteristic vectors of the linear automorphism $\varphi_1\varphi_2^{-1}$.*

Proof. We have already seen the necessity of the condition. To prove its

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sufficiency, let us write

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_r ,$$

where the M_j , $j = 1, 2, \cdots , r$, are the characteristic subspaces corresponding to the distinct characteristic roots α_j . Since $\varphi_1\varphi_2^{-1}$ is an automorphism $\alpha_j \neq 0$, $j = 1, 2, \cdots , r$. We are going to show now that this is a sum of orthogonal subspaces with respect to both forms. Thus, let $y_h \in M_h$ and $z_k \in M_k$, then

$$(y_h , z_k)_1 = \langle y_h\varphi_1 , z_k\rangle = (y_h\varphi_1\varphi_2^{-1}, z_k)_2 = \alpha_h(y_h , z_k)_2 .$$

Similarly

$$(z_k , y_h)_1 = \alpha_k(z_k , y_h)_2 ,$$

and since the forms are symmetric we get

$$\alpha_h(y_h , z_k)_2 = \alpha_k(y_h , z_k)_2 .$$

Hence, if $h \neq k$, $(y_h , z_k)_2 = 0$ and $(y_h , z_k)_1 = 0$. When $h = k$ we get $(y_h , z_h)_1 = \alpha_h(y_h , z_h)_2$ and therefore any orthogonal basis of M_h with respect to one form is also an orthogonal basis relative to the other. It is well-known that if the characteristic of the base field is $\neq 2$ any symmetric bilinear form has an orthogonal basis, so the theorem is proved.

2. We drop now the assumption that the forms are non-degenerate but we still assume that they can be simultaneously diagonalized. So, let $x_1 , x_2 , \cdots , x_n$ be an orthogonal basis with respect to both forms. By reordering the indices we can assume that

(i) $(x_i , x_i)_1 = (x_i , x_i)_2 = 0$ for $i \leqq r$,
(ii) $(x_i , x_i)_1 = 0$, $(x_i , x_i)_2 \neq 0$ for $r < i \leqq r + s$,
(iii) $(x_i , x_i)_1 \neq 0$, $(x_i , x_i)_2 = 0$ for $r + s < i \leqq r + s + t$,
(iv) $(x_i , x_i)_1 \neq 0$, $(x_i , x_i)_2 \neq 0$ for $i > r + s + t$,

Then

$$M = M_{00} \oplus M_{01} \oplus M_{10} \oplus M_{11} ,$$

where

$$M_{00} = [x_1 , \cdots , x_r], \qquad M_{01} = [x_{r+1} , \cdots , x_{r+s}],$$

$$M_{10} = [x_{r+s+1} , \cdots , x_{r+s+t}], \quad \text{and} \quad M_{11} = [x_{r+s+t+1} , \cdots , x_n],$$

is a decomposition of M as sum of orthogonal subspaces. The radical of the first form is

$$R_1 = \{y \in M \mid (y, M)_1 = 0\} = M_{00} \oplus M_{01} ,$$

the radical of the second form $R_2 = M_{00} \oplus M_{10}$. Hence the complement of $R_1 \cap R_2 = M_{00}$ in R_i is non-degenerate relative to the form $(x, y)_j$, $j \neq i$, $i = 1, 2$. So we have the following criterion.

If two symmetric bilinear forms can be simultaneously diagonalized, then the intersection $R_1 \cap R_2$ *of the radicals of the two forms has a complement in* R_1 (R_2) *which is non-degenerate relative to* $(x, y)_2((x, y)_1)$.

Now if we assume that this condition is satisfied by two given symmetric forms, we decompose M as follows. First, we find $M_{00} = R_1 \cap R_2$ and let M' be any complement of M_{00} in M; then

$$M = M_{00} \oplus M',$$

is a sum of orthogonal subspaces relative to both forms. Take $M_{01} = M' \cap R_1$, then since $M_{00} \subset R_1$, M_{01} is a complement of M_{00} in R_1 and by assumption M_{01} is non-degenerate relative to the second form, so that $M' = M_{01} \oplus M''$, where M'' is the orthogonal complement of M_{01} in M' with respect to $(x, y)_2$. Similarly, take $M_{10} = M'' \cap R_2$; this is a complement of M_{00} in R_2 since the radical of the orthogonal sum

$$M = M_{00} \oplus M_{01} \oplus M'',$$

is the direct sum of the radicals of the summands. Now we know by assumption that M_{10} is non-degenerate relative to the first form so we can find an orthogonal complement M_{11} of M_{10} in M'' relative to $(x, y)_1$. In this way we have obtained a decomposition

$$M = M_{00} \oplus M_{01} \oplus M_{10} \oplus M_{11},$$

into orthogonal subspaces. It is obvious that if the characteristic of the base field is $\neq 2$ the first three subspaces of the right hand can be simultaneously diagonalized and that the restrictions of both forms to M_{11} are non-degenerate. Therefore, to diagonalize simultaneously both forms it is enough to find a basis of M_{11} which is orthogonal relative to both forms. So we take the restriction of these forms to M_{11} and define $\varphi_1\varphi_2^{-1}$ as in section 1.

Let us remark that if N is a subspace of M_{11}, invariant under $\varphi_1\varphi_2^{-1}$, then N has the same orthogonal complement relative to both forms. For, if

$$N^{\perp_1} = \{x \in M_{11} \mid (N, x)_1 = 0\},$$

then for any $z \in N$,

$$0 = (z, N^{\perp_1})_1 = \langle z\varphi_1, N^{\perp_1}\rangle = (z\varphi_1\varphi_2^{-1}, N^{\perp_1})_2 .$$

Since $\varphi_1\varphi_2^{-1}$ is an automorphism, $N\varphi_1\varphi_2^{-1} = N$, therefore

$$N^{\perp_1} \subset N^{\perp_2},$$

but

$$\dim N^{\perp_1} = \dim M_{11} - \dim N = \dim N^{\perp_2},$$

so

$$N^{\perp_1} = N^{\perp_2}.$$

3. Let us consider now the situation in the case that both forms do not vanish simultaneously for any vector $x \neq 0$, that is, if $x \neq 0$, $(x, x)_1 = 0$ implies $(x, x)_2 \neq 0$. Then if R_1 is the radical of M relative to $(x, y)_1$, R_1 does not contain isotropic vectors relative to $(x, y)_2$; in other words $(x, x)_2 \neq 0$ for $0 \neq x \in R_2$, and consequently R_1 is non-degenerate relative to the restriction of the second form. Therefore,

$$R_1 \cap R_2 = 0 \quad \text{and} \quad M = R_1 \oplus R_2 \oplus M_{11} ,$$

that is $M_{00} = 0$, $M_{01} = R_1$, and $M_{10} = R_2$. Moreover, in the present case, the linear automorphism of $M_{11}\varphi_1\varphi_2^{-1}$ is completely reducible. For, if N is an invariant subspace of $\varphi_1\varphi_2^{-1}$, we know that

$$N^{\perp_1} = N^{\perp_2},$$

so

$$x \in N \cap N^{\perp_1},$$

implies $(x, x)_1 = 0 = (x, x)_2$, hence $x = 0$. Therefore,

$$M_{11} = N \oplus N^{\perp_1},$$

and

$$N^{\perp_1},$$

is invariant since for any

$$y \in N^{\perp_1} = N^{\perp_2},$$

we have

$$0 = (y, N)_1 = \langle y\varphi_1 , N\rangle = (y\varphi_1\varphi_2^{-1}, N)_2 ,$$

which implies

$$y\varphi_1\varphi_2^{-1} \in N^{\perp_2}.$$

Thus, any invariant subspace of $\varphi_1\varphi_2^{-1}$ has an invariant complement.

Theorem 2. *Let M be a finite dimension vector space over a real closed field. Let $(x, y)_1$ and $(x, y)_2$ be two symmetric bilinear forms on M which do not vanish simultaneously. Then, if $\dim M > 2$, the two forms can be simultaneously diagonalized.*

Proof. We have just seen that if we take an invariant subspace N of M_{11} relative to the linear automorphism $\varphi_1\varphi_2^{-1}$, then we can find an orthogonal decomposition

$$M_{11} = N \oplus N^{\perp_1}.$$

Now any irreducible polynomial with coefficient in a real closed field has degree 1 or 2, therefore we can find an orthogonal decomposition of M_{11} into subspaces of dimension 1 and 2. To complete the proof we have to show that if dim $M > 2$ the existence of an irreducible factor of degree 2 in the characteristic polynomial of $\varphi_1\varphi_2^{-1}$ is inconsistent with the assumption that the two forms do not vanish simultaneously.

Let us take a 2 dimensional space P with 2 non-degenerate symmetric bilinear forms which cannot be simultaneously diagonalized and do not vanish simultaneously. This implies that neither form can be definite. So, let x, y be a basis such that the matrices of the bilinear forms relative to this basis are

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}.$$

Since the forms do not vanish simultaneously $\alpha \neq 0$ and $\gamma \neq 0$. Take a vector $x + \delta y$ with $\delta \neq 0$. Then

$$[x + \delta y]^{\perp_1} = [x - \delta y] \quad \text{and} \quad [x + \delta y]^{\perp_2} = [(\beta + \delta\gamma)x - (\alpha + \delta\beta)y].$$

Now

$$[x + \delta y]^{\perp_1} = [x + \delta y]^{\perp_2},$$

implies that

$$x + \delta y, \qquad [x + \delta y]^{\perp_1},$$

is an orthogonal basis of P relative to both forms. Since we assume that there are not such bases, the vectors $x - \delta y$ and $(\beta + \delta\gamma)x - (\alpha + \delta\beta)y$ must be linearly independent for all $\delta \neq 0$; therefore the equation in δ

$$\alpha + \delta\beta = \delta(\beta + \delta\gamma),$$

that is,

$$\alpha = \delta^2\gamma,$$

has no solution in the real closed field, hence α and γ have different signs.

If dim $M > 2$, let z' be any non-zero vector in $P^{\perp}$. Then either $(z', z')_1 \neq 0$ or $(z', z')_2 \neq 0$. We can assume without loss of generality that $(z', z')_1 \neq 0$ and taking $-(x, y)_1$, if necessary, we can also assume that $(z', z') > 0$, so that $(z, z)_1 = 2$ for a suitable scalar multiple $z = \mu z'$. Now for any $\delta \neq 0$,

$$(z + \delta x - \delta^{-1}y, z + \delta x - \delta^{-1}y)_1 = 0, \quad \text{and}$$

$$(z + \delta x - \delta^{-1}y, z + \delta x - \delta^{-1}y)_2 = (z, z) + \delta^2\alpha + \delta^{-2}\gamma - 2\beta.$$

But the equation in δ

$$\delta^2\alpha + (z, z) - 2\beta + \delta^{-2}\gamma = 0,$$

that is,

$$\delta^4\alpha + ((z, z) - 2\beta)\delta^2 + \gamma = 0,$$

has a non-zero solution in a real closed field since α and γ have different signs. On the other hand, if $\delta \neq 0$ satisfies the equation, then

$$(z + \delta x - \delta^{-1}y, z + \delta x - \delta^{-1}y)_i = 0 \quad \text{for} \quad i = 1, 2,$$

but this is a non-zero vector, which contradicts the assumption that the forms do not vanish simultaneously. Hence, if dim $M > 2$ the minimum polynomial of $\varphi_1\varphi_2^{-1}$ is a product of linear factors and, consequently, M_{11} has a basis of characteristic vectors which gives an orthogonal basis relative to both forms.

References

[1] E. Calabi, Linear systems of real quadratic forms., *Proc. Amer. Math. Soc.*, **15** (1964) 844–46.

[2] W. Greub, *Linear algebra*, Berlin, Springer, 1958.

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