MATRIX $\aleph$-RINGS

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1. If $\mathcal{R}$ is a regular ring with unit element, the lattice $\bar{L}_{\mathcal{R}}(\bar{R}_{\mathcal{R}})$ of principal left (right) ideals ordered by inclusion is a complemented modular lattice and the lattices $\bar{L}_{\mathcal{R}}$ and $\bar{R}_{\mathcal{R}}$ are dual isomorphic, [**5**, Part II, Chapter II]. $\mathcal{R}$ is called an $\aleph$-ring if $\bar{L}_{\mathcal{R}}$ is $\aleph$-complete and $\aleph$-continuous and when $\mathcal{R}$ is an $\aleph$-ring for any cardinal number $\aleph$, $\mathcal{R}$ is a von Neumann ring.

The ring $\mathcal{R}_n$ of $n\times n$ matrices with entries in $\mathcal{R}$ is regular if $\mathcal{R}$ is regular, but the fact that $\mathcal{R}$ is an $\aleph$-ring does not guarantee that $\mathcal{R}_n$ is also an $\aleph$-ring. If $\mathcal{R}_n$ is an $\aleph$-ring (von Neumann ring) for every positive integer n, we say that $\mathcal{R}$ is a matrix $\aleph$-ring (von Neumann ring). In the present note we give a necessary and sufficient condition for an $\aleph$-ring to be a matrix $\aleph$-ring, and two examples of matrix $\aleph$-rings.

As a consequence of the additivity of upper $\aleph$-continuity in $\aleph$-complete, complemented modular lattices (see [**1**, Theorem 4.3]) and the additivity of $\aleph$-completeness under certain conditions it was shown in [**2**] that $\mathcal{R}$ is a matrix $\aleph$-ring if $\mathcal{R}_2$ is an $\aleph$-ring [**2**, Corollary 3 of Theorem 3.1].

2. In what follows $\mathcal{R}$ denotes a regular ring with unit element, $(u)_l$ and $(u)_r$ are the principal left and right ideal, respectively, generated by $u\in\mathcal{R}$. Ω denotes an ordinal number and $\bar{\Omega}$ its cardinal power.

It is convenient to think of $\bar{L}_{\mathcal{R}_2}$ as the lattice of finitely generated submodules of the left $\mathcal{R}$-module of ordered pairs (a_1, a_2), $a_i\in\mathcal{R}$, [**5**, Part II, Chapter II, Appendix 3]. $\{(a_1, a_2)\}$ will denote the submodule generated by (a_1, a_2). A finitely generated submodule M of the left $\mathcal{R}$-module of ordered pairs admits a canonical basis, that is,

$$M = \{(e, 0)\} \oplus \{(a, f)\},$$

where $e^2=e$, $f^2=f$, $fa=a$, $ae=0$ and $\oplus$ means direct sum. The submodule $\{(e, 0)\}$ is uniquely defined by M since it is equal to $M\cap\{(1, 0)\}$, that is, $\{(e, 0)\}$ is the submodule of elements of M whose second component is zero.

Our first step is to find a decomposition of M where the submodule $M\cap\{(0, 1)\}$ also appears explicitly.

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LEMMA 1. *Any finitely generated submodule M of the left $\mathfrak{R}$-module of ordered pairs (a_1, a_2), $a_i \in \mathfrak{R}$, can be decomposed in the following way*

$$M = \{(e_1, 0)\} \oplus \{(a_1, a_2)\} \oplus \{(0, e_2)\}$$

where $e_i^2 = e_i$, $a_1e_1 = a_2e_2 = 0$ and $(a_1)_r = (a_2)_r$.

PROOF. Let U be the involutorial automorphism of the left $\mathfrak{R}$-module of ordered pairs which takes the vector (a, b) into (b, a). U takes a finitely generated submodule into a finitely generated submodule and defines an involutorial automorphism $\overline{U}$ of $\overline{L}_{\mathfrak{R}_2}$.

Suppose $M = \{(e_1, 0)\} \oplus \{(a, f)\}$ under a canonical decomposition. Then $e_1a = 0$, $fa = a$. Taking a canonical decomposition of

$$\overline{U}\{(a, f)\} = \{(f, a)\} = \{(e_2, 0)\} \oplus \{(a_2, a_1)\},$$

$e_2^2 = e_2$, $a_2e_2 = 0$ and $(a_1)_l = (a)_l$. Therefore

$$M = \{(e_1, 0)\} \oplus \{(a_1, a_2)\} \oplus \{(0, e_2)\}.$$

Since $M \cap \{(0, 1)\} = \{(a, f)\} \cap \{(0, 1)\}$, it follows that $M \cap \{(0, 1)\} = \{(0, e_2)\}$. Because $a_1 = xa$ and $ae_1 = 0$, we have $a_1e_1 = 0$. Moreover, $\{(a_1, a_2)\} \cap \{(0, 1)\} = \{(a_1, a_2)\} \cap \{(1, 0)\} = 0$ implies that the left annihilators of a_1 and a_2 coincide, hence $(a_1)_r = (a_2)_r$.

3. The next point which we need to emphasize is the fact that a left factor-correspondence ([5, Part II, Definition 15.1], notice that this definition of f.-c. is for right principal ideals which we call right factor-correspondence) between $(u)_l$ and $(v)_l$ can be determined by a pair of elements of $\mathfrak{R}$. This fact is an immediate consequence of the definition of factor-correspondence and its precise statement is given in the following lemma.

LEMMA 2. *A left factor-correspondence between $(u)_l$ and $(v)_l$ is completely determined by any pair of elements $u', v' \in \mathfrak{R}$ corresponding to each other and such that $(u')_l = (u)_l$ and $(v')_l = (v)_l$. Conversely, if $(u')_r = (v')_r$, the one-to-one mapping*

$$xu' \leftrightarrow xv', \qquad x \in \mathfrak{R},$$

defines a factor-correspondence between $(u')_l$ and $(v')_l$.

The factor-correspondence defined by the pair u, v will be denoted by $(u:v)$.

We introduce an order in the set of factor-correspondences by defining

$$(u_1:v_1) \geqq (u_2:v_2)$$

if $(u_1)_l \supset (u_2)_l$ and u_2 and v_2 correspond to each other in $(u_1:v_1)$.

THEOREM 1. *Let* $\mathfrak{R}$ *be a regular ring such that* $\bar{L}_{\mathfrak{R}}$ *is upper* $\aleph$*-complete and upper* $\aleph$*-continuous. Then the lattice* $\bar{L}_{\mathfrak{R}_2}$ *is upper* $\aleph$*-complete if and only if every increasing chain* $(u^\alpha : v^\alpha)$, $\alpha < \Omega$ *and* $\bar{\Omega} \leqq \aleph$, *of left factor-correspondences has a supremum. Moreover, if* $\bar{L}_{\mathfrak{R}_2}$ *is* $\aleph$*-complete it is upper* $\aleph$*-continuous.*

PROOF. The last statement is an immediate consequence of the theorem of Amemiya and Halperin on the additivity of upper $\aleph$-continuity is complemented, $\aleph$-complete modular lattices, (cf. [1, Theorem 4.3]). For, $\bar{L}_{\mathfrak{R}_2} = [0, \{(1, 0)\} \cup \{(0, 1)\}]$ and $[0, \{(1, 0)\}]$ is isomorphic to $\bar{L}_{\mathfrak{R}}$.

Assume that $\bar{L}_{\mathfrak{R}_2}$ is upper $\aleph$-complete and let $(u^\alpha : v^\alpha)$, $\alpha < \Omega$ and $\bar{\Omega} \leqq \aleph$, be an increasing chain of left factor-correspondences. Then the modules $M^\alpha = \{(u^\alpha, v^\alpha)\}$ form an increasing Ω-chain. Since $M^\alpha \cap \{(1, 0)\} = M^\alpha \cap \{(0, 1)\} = 0$, $(\bigcup(M^\alpha | \alpha < \Omega)) \cap \{(0, 1)\} = (\bigcup(M^\alpha | \alpha < \Omega)) \cap \{(1, 0)\} = 0$, because $\bar{L}_{\mathfrak{R}_2}$ is upper $\aleph$-continuous. Therefore, by Lemma 1, $\bigcup(M^\alpha | \alpha < \Omega) = \{(u, v)\}$ with $(u)_r = (v)_r$ and it is clear that $(u : v)$ is the supremum of the $(u^\alpha : v^\alpha)$.

So we assume now that, if $\bar{\Omega} \leqq \aleph$, every increasing Ω-chain of left factor-correspondences has a supremum and proceed to show that $\bar{L}_{\mathfrak{R}_2}$ is upper $\aleph$-complete. Let M^α be an increasing Ω-chain of modules in $\bar{L}_{\mathfrak{R}_2}$ and let

$$M^\alpha = \{(e^\alpha, 0)\} \oplus \{(a^\alpha, f^\alpha)\}$$

be a canonical decomposition of the M^α. If $(e_1)_l = \bigcup((e^\alpha)_l | \alpha < \Omega)$, it is clear that the chain M^α has a supremum if and only if the increasing chain of $N^\alpha = M^\alpha \cup \{(e_1, 0)\}$ has a supremum. Now,

$$N^\alpha = \{(e_1, 0)\} \oplus \{(b^\alpha, f^\alpha)\}$$

with $b^\alpha = a^\alpha - a^\alpha e_1$, is a canonical decomposition of N^α and we only need show that the chain of modules $\{(b^\alpha, f^\alpha)\}$ has a supremum. This chain is actually increasing, for $\{(b^\alpha, f^\alpha)\} \subset N^\beta$ for $\alpha \leqq \beta$ implies that

$$(b^\alpha, f^\alpha) = d_1(e_1, 0) + d_2(b^\beta, f^\beta) = (d_1 e_1 + d_2 b^\beta, d_2 f^\beta),$$

hence $b^\alpha = d_2 b^\beta$, because $b^\gamma e_1 = 0$ for all $\gamma < \Omega$, and consequently $(b^\alpha, f^\alpha) = d_2(b^\beta, f^\beta)$. By Lemma 1,

$$\{(b^\alpha, f^\alpha)\} = \{(b_1^\alpha, b_2^\alpha)\} \oplus \{(0, e_2^\alpha)\}, \tag{1}$$

where $b_2^\alpha e_2^\alpha = 0$ and $(b_1^\alpha)_r = (b_2^\alpha)_r$. Now $\{(b^\beta, f^\beta)\} \cap \{(1, 0)\} = 0$ implies $(b_2^\alpha)_l \cap (e_2^\beta)_l = 0$ if $\alpha < \beta$; for it follows from $d_1 b_2^\alpha = d_2 e_2^\beta$ that

$$(d_1 b_1^\alpha, 0) = d_1(b_1^\alpha, b_2^\alpha) - d_2(0, e_2^\beta) \subset \{(b^\beta, f^\beta)\},$$

hence $d_1b_1^\alpha=0$ and consequently $d_1b_2^\alpha=0$, since $(b_1^\alpha)_r=(b_2^\alpha)_r$. If $(e_2)_l=\bigcup((e_2^\alpha)_l\mid\alpha<\Omega)$, then, by the upper $\aleph$-continuity of $\overline{L}_{\mathfrak{R}}$,

$$(2) \qquad (e_2)_l \cap (b_2^\alpha)_l = 0.$$

The increasing chain defined by the modules (1) has a supremum if and only if the increasing chain defined by

$$P^\alpha = \{(b_1^\alpha, b_2^\alpha)\} + \{(0, e_2)\}$$

has a supremum. Now (2) implies $P^\alpha \cap \{(1, 0)\}=0$, hence, by Lemma 1,

$$P^\alpha = \{(a_1^\alpha, a_2^\alpha)\} \oplus \{(0, e_2)\},$$

where $a_2^\alpha e_2=0$ and $(a_1^\alpha)_r=(a_2^\alpha)_r$. It is easily seen that the modules $\{(a_1^\alpha, a_2^\alpha)\}$, $\alpha<\Omega$, form an increasing chain; therefore $(a_1^\alpha:a_2^\alpha)$ is an increasing chain of left factor-correspondences. Let $(a_1:a_2)$ be the supremum of this chain, then

$$\{(a_1, a_2)\} \oplus \{(0, e_2)\} = \bigcup(P^\alpha \mid \alpha < \Omega) = \bigcup(\{(b^\alpha, f^\alpha)\} \mid \alpha \in \Omega).$$

THEOREM 2. *Let $\mathfrak{R}$ be an $\aleph$-ring. Then $\mathfrak{R}_2$ is an $\aleph$-ring if and only if every increasing Ω-chain of left or right factor-correspondences has a supremum when $\overline{\Omega}\leqq\aleph$.*

PROOF. This is a consequence of Theorem 1 and the dual isomorphism between $\overline{L}_{\mathfrak{R}_2}$ and $\overline{R}_{\mathfrak{R}_2}$.

COROLLARY 1. *Any complete rank-ring $\mathfrak{R}$ is a matrix von Neumann ring.* (*See* [5, *Part* II, *Definition* 18.1].)

PROOF. Given any increasing chain of left factor-correspondences $(u^\alpha:v^\alpha)$, $\alpha<\Omega$, the u^α can be chosen to be idempotents. Let $\overline{R}$ denote the rank-function. Since $\overline{R}(u^\beta)$ is a bounded increasing chain of real numbers, we can replace the given chain by an increasing sequence $(u^{\alpha_i}: v^{\alpha_i})$, $i=1, 2, \cdots$, with the same upper bound. Then we can even assume that the u^{α_i} besides being idempotent satisfy $u^{\alpha_i}u^{\alpha_j}=u^{\alpha_j}u^{\alpha_i}=u^{\alpha_i}$ for $i<j$ (we only have to apply the construction in the proof of Lemma 18.3 of [5] to the sequence $1-u^{\alpha_i}$). Now, since $(u^{\alpha_j}-u^{\alpha_i})u^{\alpha_i}=u^{\alpha_i}(u^{\alpha_j}-u^{\alpha_i})=0$ and $(u^{\alpha_j}-u^{\alpha_i})^2=u^{\alpha_j}-u^{\alpha_i}$,

$$\overline{R}(u^{\alpha_j} - u^{\alpha_i}) = \overline{R}(u^{\alpha_j}) - \overline{R}(u^{\alpha_i}).$$

Therefore, by the completeness of $\mathfrak{R}$, the Cauchy sequence u^{α_i}, $i=1, 2, \cdots$ has a limit u. On the other hand, $\overline{R}(v^{\alpha_j}-v^{\alpha_i})=\overline{R}(u^{\alpha_j}-u^{\alpha_i})$ since $v^{\alpha_j}-v^{\alpha_i}$ and $u^{\alpha_j}-u^{\alpha_i}$ correspond to each other under $(u^{\alpha_j}:v^{\alpha_j})$. Then, if $\lim_{i\to\infty} v^{\alpha_i}=v$, $(u:v)$ is the supremum of the given chain.

COROLLARY 2. *If $\mathfrak{R}$ is an $\aleph$-ring and $\bar{L}_{\mathfrak{R}}$ has a large 2 basis, then every increasing chain of left or right factor-correspondences in $\mathfrak{R}$ has a supremum.*

PROOF. The corollary follows from Theorem 2 and [**2**, Corollary 2 of Theorem 3.1].

4. As an application of Theorem 2 we give two examples of matrix $\aleph$-rings.

EXAMPLE 1. (This is a generalization of Kaplansky's example [**3**, p. 526] and [**4**, Example 3, p. 604].) Let J be any set such that $\bar{J} \geqq \aleph$. Let $\{D_\alpha\}_{\alpha\in J}$ be a family of division rings and F_α a proper division subring of D_α for every $\alpha \in J$. Consider the functions f which map each element $\alpha \in J$ into an element of D_α and such that, if $J_f = \{\alpha \mid \alpha\in J, f(\alpha) \notin F_\alpha\}$, $\bar{J}_f \leqq \aleph$. Then the ring $\mathfrak{R}$ of such functions under the natural definition of addition and multiplication is a von Neumann ring. Applying Theorem 2 it is easily seen that $\mathfrak{R}$ is a matrix $\aleph$-ring, but, if $\bar{J} > \aleph$, $\mathfrak{R}$ is not a matrix $\aleph'$-ring for any $\aleph' > \aleph$.

EXAMPLE 2. Let $\mathfrak{B}$ be an $\aleph$-complete Boolean algebra and X its dual space, that is, X is the space of the Stone representation. Then X is a totally disconnected, compact, Hausdorff space. Consider the functions f over X with values in a Galois field F satisfying the condition: for every $a \in F$, the set

$$X_a = \{x \mid x \in X, f(x) = a\}$$

is a clopen set of X. Then, under the natural definition of addition and multiplication of a function, such functions form a matrix $\aleph$-ring.

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