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THE AUTOMORPHISMS OF $PO_8^+(Q)$ AND $PS_8^+(Q)$.*

By MARÍA J. WONENBURGER.[1]

The purpose of the present paper is to complete some results of [8] by studying the automorphisms of $PS_8^+(Q)$, the projective group of proper similitudes of an 8-dimensional vector space M with respect to a non-degenerate quadratic form Q, and of its subgroup $PO_8^+(Q)$, the projective group of rotations. It is well-known that in certain cases these groups actually have exceptional automorphisms, that is, automorphisms which are not induced by automorphisms of $S_8^+(Q)$, the group of proper similitudes.

We prove, under the assumption that the base field K has characteristic not 2, that $PS_8^+(Q)$ has exceptional automorphisms if, and only if, either the quadratic form Q or a scalar multiple αQ permits composition. As for $PO_8^+(Q)$, it has exceptional automorphisms if, and only if, the vector space M has an orthonormal basis with respect to either Q or a multiple αQ, and the base field is Pythagorean; this implies $PO_8^+(Q) \cong PS_8^+(Q)$.

In the last section we show that, if $PS_8^+(Q)$ has exceptional automorphisms, any exceptional one together with the ones induced by the automorphisms of $S_8^+(Q)$ generate the whole group of automorphisms.

1. We are going to use the notation of [8, Sections 1, 5] with the only difference that now Q *is always a non-degenerate quadratic form over an 8-dimensional vector space* M, and we assume again that K has characteristic not 2. Moreover, since two quadratic forms which are obtained one from the other by scalar multiplication define the same group of similitudes and related groups we can assume in our arguments without loss of generality that there exists a vector $x \in M$ such that $Q(x) = 1$.

THEOREM 1. *The group $PS^+(Q)$ has exceptional automorphisms if, and only if, the quadratic form Q (up to a scalar factor) permits composition* (see e.g. [4]).

Proof. If an automorphism ϕ of $PS^+(Q)$ takes any $(6,2)$ coset into another $(6,2)$ coset, then the restriction of ϕ to $PO^+(Q)$ is an automorphism of $PO^+(Q)$ induced by an automorphism of $O^+(Q)$ (see [2, Section 37]) and

* Received July 9, 1962.

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therefore ϕ is an automorphism of $PS^{\cdot}(Q)$ induced by an automorphism of $S^{\cdot}(Q)$ (see the proof of [8, Lemma 2]).

We consider now two different cases.

Case 1. *K has more than 5 elements.* Then ϕ is an exceptional automorphism if, and only if, there exists a $(6,2)$ coset $\bar{U}$ which is not taken into a $(6,2)$ coset and therefore by [8, Corollary of Lemma 3] $\phi(\bar{U})$ is the coset of a P-involution. Let x_1, x_2 be any orthogonal basis for the minus-space of U and $x_3, x_4, \cdots, x_8$ any orthogonal basis of its plus-space. Let U_i, $i = 2, 3, \cdots, 8$, be the $(6,2)$ involutions whose minus-spaces are spanned by x_1 and x_i, so that $\bar{U}_2 = \bar{U}$. If $\phi(\bar{U}_i) = \bar{T}_i$, in the course of the proof of [8, Lemma 6] it was shown that the T_i, $i = 2, 3, \cdots, 8$, anticommute with each other and, since the $\bar{T}_i$ are either $(6,2)$ cosets or cosets of P-involutions, the only possibility is that the T_i are anticommuting P-involutions.

Assume now that ρ and ρ' are the ratios of any two of these P-involutions, say T_2 and T_3, and let x be any vector of M such that $Q(x) = 1$. The vectors x, xT_2, xT_3, xT_2T_3, are mutually orthogonal non-isotropic vectors (cf. [8, proof of Lemma 4]). Moreover, if N is the 4-dimensional space spanned by these vectors, N is invariant under the P-involutions T_2 and T_3; hence $N^{\perp}$, the orthogonal complement of N, is also invariant under T_2 and T_3. Now, if $y \in N^{\perp}$ and $Q(y) = \beta \neq 0$, y, yT_2, yT_3, yT_2T_3 is an orthogonal basis of $N^{\perp}$. We have found, then, an orthogonal basis, x, xT_2, xT_3, xT_2T_3, y, yT_2, yT_3, yT_2T_3, of M such that the lengths $Q(\ )$ of its vectors are

$$1,\ \rho,\ \rho',\ \rho\rho',\ \beta,\ \beta\rho,\ \beta\rho',\ \beta\rho\rho',$$

respectively, which shows that the quadratic form Q permits composition (see [4, Sections 1, 2]). Hence, if $PS^{\cdot}(Q)$ has exceptional automorphisms, the quadratic form Q (up to a scalar factor) is the quadratic form associated to a Cayley algebra.

On the other hand, it is mentioned in [1] that if Q is the quadratic form associated to a Cayley algebra, the automorphisms ϕ_1 and ϕ_2 of $PS^{\cdot}(Q)$ defined in [1, Cor. 2 of Th. 1] are exceptional. (In the proof of Th. 3 below we show that the automorphism ϕ_2 is exceptional.)

Case 2. *K is a finite field.* If the quadratic form Q has square discriminant, Q is the quadratic form of a split Cayley algebra and the automorphisms ϕ_2 of [1, Cor. 2 of Th. 1] is exceptional. When the discriminant of Q is not a square Q does not permit composition and the commutator group of $PS^{\cdot}(Q)$ is $PO^{\cdot}(Q) \cong P\Omega(Q)$. This implies that any automorphism ϕ of $PS^{\cdot}(Q)$ induces an automorphism in $PO^{\cdot}(Q)$, but the automorphisms of

$PO^{\cdot}(Q)$ are induced by automorphisms of $O^{\cdot}(Q)$ (see [8, Th. 4]) and, consequently, ϕ is induced by an automorphism of $S^{\cdot}(Q)$.

LEMMA 1. *If the group $PO^{+}(Q)$ has exceptional automorphisms the space M has an orthonormal basis with respect to Q or αQ* (see [2, p. 60]).

Proof. When K is a finite field, $PO^{\cdot}(Q)$ does not have exceptional automorphisms ([8, Th. 4]). It follows from [7] that if K has more than 5 elements an automorphism of $PO^{\cdot}(Q)$ can not take a $(6,2)$ coset into a $(4,4)$ coset. (The proof is like the proof of [8, Lemma 3] using $PO^{\cdot}(Q)$ instead of $PS^{\cdot}(Q)$.) Hence, under an automorphism ϕ of $PO^{\cdot}(Q)$, a $(6,2)$ coset must go either into a $(6,2)$ coset or into the coset of an orthogonal P-involution. We have again that, if ϕ is an exceptional automorphism, there should exist a $(6,2)$ coset $\bar{U}$ which is taken by ϕ into the coset of an orthogonal P-involution, that is, a P-involution of ratio 1. Define $x_1, x_2, x_3, \cdots, x_8$ and the $\bar{U}_i$ as before and let $\phi(\bar{U}_i) = \bar{T}_i$. Then [8, Lemma 6] still applies and we can conclude that the T_i, $i = 2, 3, \cdots, 8$, are seven orthogonal P-involutions such that any two of them anticommute. If x is any vector with $Q(x) = 1$, then x together with the vectors xT_i, $i = 2, 3, \cdots, 8$ forms an orthonormal basis of M.

LEMMA 2. *If an automorphism of $PO^{+}(Q)$ (or $PS^{+}(Q)$, if K has more than 5 elements) is exceptional it takes every $(6,2)$ coset into the coset of a P-involution.*

Proof. If K is a finite field, $PO^{\cdot}(Q)$ has no exceptional automorphism and there is nothing to prove. Hence we can assume that K has more than 5 elements. Suppose then that the automorphism ϕ takes the coset of a $(6,2)$ involution U into the coset of a P-involution. We have already shown that if x_1, x_2 and $x_3, \cdots, x_8$ are any orthogonal bases of the minus-space M_1 and plus-space $M_1^{\perp}$, respectively, of U, the cosets defined by the $(6,2)$ involutions $\bar{U}_i$, $i = 2, \cdots, 8$, are taken into the cosets of anticommuting P-involutions T_i. Now, if U_{ij} is the involution whose minus-space is spanned by x_i and x_j, $i, j \neq 1$, $U_{ij} = U_iU_j$, hence $\phi(\bar{U}_{ij}) = \bar{T}_i\bar{T}_j$ and, as a consequence of $T_iT_j = -T_jT_i$, T_iT_j is also a P-involution. This implies that any $(6,2)$ coset whose minus space is orthogonal to M_1 is mapped by ϕ into a P-involution.

Now, let N be the minus space of any $(6,2)$ involution V. The space $(M_1 + N)^{\perp}$ has dimension $\geqq 4$ and, since $(M_1 + N) \cap (M_1 + N)^{\perp}$ is at most 2 dimensional, $(M_1 + N)^{\perp}$ contains a 2-dimensional non-isotropic subspace R. Since M_1 is orthogonal to R, ϕ maps the coset of the $(6,2)$ involution with minus-space R into the coset of a P-involution, and therefore ϕ also maps $\bar{V}$ into the coset of a P-involution because R if orthogonal to N.

LEMMA 3. *If the vector space M has an orthonormal basis with respect to Q, then any orthogonal P-involution belongs to the commutator group of $O^+(Q)$.*

Proof. Since any P-involution takes any vector x into a vector orthogonal to x, it is an immediate consequence of Witt's theorem that given an orthogonal P-involution T it is possible to choose an orthonormal basis of M of the form $x_i, y_i = x_iT$, $i = 1, 2, 3, 4$ (cf. the proof of Lemma 4 below). Define the linear transformations T_1 and S as follows,

$$x_jT_1 = y_j, \quad y_jT_1 = -x_j, \quad \text{for } j = 1, 2$$
$$x_jT_1 = x_j, \quad y_jT_1 = y_j, \quad \text{for } j = 3, 4, \text{ and}$$
$$x_iS = y_{i+2} \quad y_iS = x_{i+2}. \quad \text{where the indexes } i + 2 \text{ should}$$

be computed modulo 4.

Then T_1 and S are rotations and $T = T_1ST_1^{-1}S_1^{-1}$.

THEOREM 2. *The group $PO^+(Q)$ has exceptional automorphisms if, and only if, M has an orthonormal basis with respect to Q or αQ and the base field K is Pythagorean.*

Proof. If $PO^+(Q)$ has exceptional automorphisms, let x_i, $i = 1, 2, \cdots, 8$ be an orthonormal basis of M with respect to Q or αQ. If K is not Pythagorean, assume that $\alpha^2 + \beta^2$ is not a square and let U be the $(6, 2)$ involution whose minus-space is spanned by x_1 and $\alpha x_2 + \beta x_3$. Then the coset of U does not belong to $P\Omega(Q)$ since the spin-norms of U and $-U$ belong to the quadratic class of $\alpha^2 + \beta^2$ which is not a square. Hence $\bar{U}$ can not be mapped by an automorphism of $PO^+(Q)$ into the coset of an orthogonal P-involution. Therefore, by Lemma 2, $PO^+(Q)$ has no exceptional automorphism. This proves the "only if" part of the theorem.

On the other hand, if the conditions of the theorem are satisfied, the ratio of any similitude is a square and, consequently, $PO^+(Q) \cong PS^+(Q)$. Hence the "if part" of the theorem follows from Theorem 1.

COROLLARY. *If $PO^+(Q)$ has exceptional automorphisms, then $PO^+(Q) \cong PS^+(Q)$.*

2. In this section we are going to determine the form of all the automorphisms of $PS^+(Q)$. By the Corollary that we have just established, this gives us also the automorphisms of $PO^+(Q)$.

LEMMA 4. *In $PS^+(Q)$ the cosets of P-involutions of the same ratio decompose into two conjugate classes.*

Proof. We will prove first that in $S(Q)$ any two P-involutions T_1 and T_2 of ratio ρ are conjugate.

Let x_i, x_iT_1, $i = 1, 2, 3, 4$, be an orthogonal basis of M with respect to Q. Let y_1 be any vector such that $Q(y_1) = Q(x_1)$. Then the 2-dimensional subspaces N_1 and M_1 spanned by y_1 and y_1T_2, and x_1 and x_1T_1, respectively, are isometric. Hence, by Witt's theorem, $N_1^{\perp}$ and $M_1^{\perp}$ are also isometric and therefore there exists a vector $y_2 \in N_1^{\perp}$ such that $Q(y_2) = Q(x_2)$. Now the subspaces N_2 and M_2 spanned by y_i, y_iT_2, $i = 1, 2$, and x_i, x_iT_1, $i = 1, 2$, respectively, are isometric and we can repeat the preceding process until we get an orthogonal basis y_i, y_iT_2, $i = 1, 2, 3, 4$ such that $Q(x_i) = Q(y_i)$. Define the linear transformation U by

$$x_iU = y_i, \qquad (x_iT_1)U = y_iT_2, \qquad i = 1, 2, 3, 4.$$

Then U is an orthogonal transformation and $T_2 = U^{-1}T_1U$.

$PS(Q)$ consists of the subgroup $PS^+(Q)$ and the cosets $PS^-(Q)$, therefore in $PS^+(Q)$ the cosets defined by all the P-involutions of ratio ρ are either all conjugate or they decompose into the two conjugate classes. But, if $V \in S^-(Q)$, the coset defined by VTV^{-1} can not be equal to a coset defined by WTW^{-1} with $W \in S^+(Q)$, because this would imply

$$(W^{-1}V)T(W^{-1}V)^{-1} = \pm T, \qquad W^{-1}V \in S^-(Q),$$

and there are no improper similitudes commuting or anticommuting with T (see [6, Prop. 3 and 4]). This proves the lemma.

THEOREM 3. *If the quadratic form Q permits composition, then the group of automorphisms of $PS^+(Q)$ is generated by any exceptional automorphism and the ones induced by automorphisms of $S^+(Q)$.*

Proof. We are going to show first that the automorphism ϕ_2 of $PS^+(Q)$ defined in [1, Corollary 2 of Th. 1] is exceptional. Let e be the element of M which acts as the identity of the Cayley algebra and let b be any non-isotropic vector orthogonal to e. The proof of the corollary mentioned above shows that ϕ_2 takes the coset of the $(6, 2)$ involution whose minus space is spanned by e and b into the coset of the transformation R_b defined by right multiplication by b. Hence R_b has ratio $Q(b)$ and $R_b{}^2$ is the multiplication by $-Q(b)$. This shows that R_b is a P-involution of ratio $Q(b)$ and ϕ_2 is an exceptional automorphism.

If there exists in $PS^+(Q)$ a P-involution T of ratio ρ, since eT is orthogonal to e and $Q(eT)=\rho$, the coset of the $(6,2)$ involution U whose minus-space is spanned by e and eT is taken by ϕ_2 into the coset of the P-involution R_{eT} of ratio ρ. Moreover the $(6,2)$ cosets conjugate to $\bar{U}$ are mapped by ϕ_2 into cosets conjugate to $\bar{R}_{eT}$ in $PS^+(Q)$.

We consider again two different cases.

Case 1. *K has more than 5 elements.* Let ψ be any exceptional automorphism of $PS^+(Q)$, $\bar{U}$ any $(6,2)$ coset and $\psi(\bar{U})=\bar{T}$. Then, by Lemma 2, we know that T is a P-involution and we have two possibilities,

a) $\bar{T}$ is conjugate to $\bar{R}_{eT}$. Then $\phi_2{}^{-1}\psi$ takes $\bar{U}$ into another $(6,2)$ coset and therefore is an automorphism induced by an automorphism of $S^+(Q)$, that is, $\phi_2{}^{-1}\psi=\sigma_A$, where $\sigma_A(\bar{X})=\overline{AXA^{-1}}$ and A is a semi-similitude, and $\psi=\phi_2\sigma_A$,

b) $\bar{T}$ is not conjugate to $\bar{R}_{eT}$. Then, if $V\in S^-(Q)$, $\overline{VTV^{-1}}$ is conjugate to $\bar{R}_{eT}$ and consequently σ_V, where σ_V is the automorphism of $PS^+(Q)$ defined by $\sigma_V(\bar{X})=\overline{VXV^{-1}}$, is of the form $\sigma_V\psi=\phi_2\sigma_A$, that is, $\psi=\sigma_{V^{-1}}\phi_2\sigma_A$.

We have shown then that ϕ_2 and the automorphisms of the form σ_A generate the whole group of automorphisms and that the group generated by any exceptional automorphism ψ and the σ_A contains ϕ_2 and therefore is the whole group of automorphisms.

Case 2. *K is a finite field.* Then the second commutator group of $PS^+(Q)$ coincides with $P\Omega(Q)$, the commutator group of $PO^+(Q)$ (actually $P\Omega(Q)$ is the first commutator group of $PS^+(Q)$ by [5, Th. 5]). Hence any automorphism ϕ of $PS^+(Q)$ induces an automorphism in $P\Omega(Q)$ and it is known that in $P\Omega(Q)$ the $(6,2)$ cosets can not be taken into $(4,4)$ cosets (see [2, Section 43]) and therefore ϕ takes $(6,2)$ cosets either into $(6,2)$ cosets or into cosets of P-involutions. But the $(6,2)$ involutions whose cosets belong to $P\Omega(Q)$ are those whose minus-space has square discriminant with respect to the restriction of Q to it, and all such involutions are conjugate in $P\Omega(Q)$. Hence we have two possibilities, under ϕ

a) either all the $(6,2)$ cosets of $P\Omega(Q)$ go into $(6,2)$ cosets, in which case the restriction of ϕ to $P\Omega(Q)$ is an automorphism induced by an automorphism of $O^+(Q)$ (see [2, Sections 43, 42]) and then, as in the proof of [8, Lemma 2], we can conclude that ϕ itself is induced by an automorphism of $S^+(Q)$.

b) all the $(6,2)$ cosets of $P\Omega(Q)$ go into P-involutions. Then, as in case 1, we can deduce that either $\phi_2{}^{-1}\phi$ or $\phi_2{}^{-1}\sigma_V\phi$ is an automorphism induced by an automorphism of $S^+(Q)$, which proves the theorem.

Remark. To carry out some proofs we have considered two different cases. This is due to the fact that [8, Cor. of Lemma 3] was established under the condition that K has more than 5 elements, but, in fact, the corollary can be easily proved when K is the field with 5 elements. We have not proved it here because we would still need a different proof for the case of a field with 3 elements.

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[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Société Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

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