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CONTENTS

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THE AUTOMORPHISMS OF THE GROUP OF SIMILITUDES AND SOME RELATED GROUPS.*[1]

By María J. Wonenburger.[2]

To the Memory of Dn. Julio Rey Pastor.

In [5] J. Dieudonné has determined the automorphisms of the orthogonal group, the group of rotations and their corresponding projective groups under the assumptions that the base field has characteristic not 2, the dimension of the vector space is sufficiently large and the index of the quadratic form is greater than zero. It has been proved that his results hold true without the restriction on the index of the quadratic form. For the case of the orthogonal group this was shown by C. Rickart in [8], for the projective group it was proved by J. Walter in [9] and for the case of the group of rotations and the projective group of rotations it was proved in [13]. Our purpose now is to determine the automorphisms of the group of similitudes, the group of proper similitudes and their corresponding projective groups. The general method consists in a reduction to Dieudonné's case, namely, it is shown first that, if an automorphism of any of the groups under consideration takes the subgroup of rotations or the cosets defined by the subgroup of rotations into themselves, the automorphisms of these groups are determined by Dieudonné's results (see Lemma 2 below and its corollaries). Hence our task is to prove the invariance of the group of rotations or the projective group of rotations under the automorphisms of the groups under consideration.

The case of the group of similitudes or the group of proper similitudes is settled first. If we deal with an odd dimensional vector space the projective group of similitudes is isomorphic to $O^+(Q)$, so there is nothing to prove. When the vector space is even dimensional the study of the automorphisms of the projective group of similitudes and the projective group of proper similitudes requires a more careful investigation of the new involutions appearing in these groups in order to characterize the cosets of the $(n-2,2)$ involutions.

* Received May 22, 1962.

[1] Most of the results in this paper are generalizations of results included in the author's dissertation written under the direction of Professor N. Jacobson and presented to Yale University in 1957.

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Once these cosets have been characterized the results follow from the corollaries of Lemma 2. Since our characterization of the cosets of $(n-2,2)$ involutions excludes the case when K has 3 or 5 elements, we study first the automorphisms of the projective groups under consideration when the base field is a finite field.

1. Let M be a left vector space over a commutative field K and Q a non-degenerate quadratic form on M. *It is always assumed that the field K has characteristic not 2.* The elements of K will be always denoted by small greek letters, the elements of M by small latin letters and the semi-linear transformations of M by capital latin letters. The image of $x \in M$ under a transformation T will be denoted by xT, and, when the scalar multiplication by $\alpha \in K$ is considered as a linear transformation it is denoted by α_L.

A semi-linear transformation T of M relative to the automorphism σ of K is called a semi-similitude if for any $x \in M$

$$Q(xT) = \rho(Q)x))^{\sigma},$$

where $\rho \neq 0$ is a fixed element of K which depends on T and is called the ratio of T. In particular, if σ is the identity, T is called a similitude. When M is odd dimensional every similitude is the scalar multiple of a rotation. When M is even dimensional, say $\dim M = 2m$, the determinant of the matrix representing the similitude T of ratio ρ with respect to any basis is either ρ^m or $-\rho^m$. In the first case T is called a proper similitude and when the determinant is $-\rho^m$ T is an improper similitude.

The similitudes form a group, which we will denote by $S(Q)$; when the space M is even dimensional this group contains as a normal subgroup of index 2 the group of proper similitudes, which we denote by $S^+(Q)$. The centres of these groups consist of the scalar multiplications with the only exception of the case of the group of proper similitudes of a two dimensional space (see e.g. [3]).

The similitudes of ratio 1 form the orthogonal group $O(Q)$ and the group of proper similitudes of ratio 1 is the group of rotations $O^+(Q)$. The centre of $O(Q)$ consists of $\pm 1_L$ with the only exception of the case when $\dim M = 2$, K is the prime field with 3 elements and Q has index 1 (see e.g. [1, page 132]).

Let K' be the multiplicative group of the field K. Then the projective groups $PS(Q)$ and $PS^+(Q)$ are defined as

$$PS(Q) = S(Q)/K', \qquad PS^+(Q) = S^+(Q)/K',$$

which, with the exception of the case of the group $S^+(Q)$ of a two dimensional space, are the factor groups of $S(Q)$ and $S^+(Q)$ by their centres. Similarly, if Z is the center of $O(Q)$, $Z = \pm 1_L$, with the only exception of the 2 dimensional case mentioned above, and

$$PO(Q) = O(Q)/Z \text{ and } PO^+(Q) = O^+(Q)/Z \cap O^+(Q).$$

We are going to use the following known results:

(i) When $\dim M = n \geqq 4$ every automorphism ϕ of $O(Q)$ may be written in the form

$$\phi(G) = T^{-1}GT\chi(G) \text{ for every } G \in O(Q),$$

where $G \to \chi(G)$ is a representation of $O(Q)$ in the multiplicative group $\{1_L, -1_L\}$ and T is a semi-similitude of Q. (Cf. [5], [8] and [6]).

(ii) When $\dim M \geqq 5$ every automorphism of $O^+(Q)$ is the restriction of an automorphism of $O(Q)$. (See [5], [6] and [13]).

(iii) When $\dim M \geqq 4$ every automorphism of $PO(Q)$, the projective orthogonal group, is induced by an automorphism of $O(Q)$. (See [5], [9] and [6]).

(iv) When $\dim M \geqq 5$ but $\dim M \neq 8$ every automorphism of $PO^+(Q)$ is induced by an automorphism of $O^+(Q)$, (cf. [5], [6] and [13], and also Theorem 4 below).

(v) When $\dim M \geqq 6$ and $\neq 8$, and K is a finite field every automorphism of $P\Omega(Q)$ the projective commutator group is induced by an automorphism of $O^+(Q)$. (Cf. [5] or [6]).

Any element of order 2 of any of the groups under consideration is called an involution of that group. It is well-known that, if U is an orthogonal involution, that is, U is an orthogonal transformation and $U^2 = 1_L$, then the space M splits in a direct sum of two subspaces,

$$M = M^+ \oplus M^-,$$

such that U leaves invariant every element of M^+ and takes every vector of M^- into its negative. The subspaces M^+ and M^- are non-isotropic and mutually orthogonal. If $\dim M^+ = p$, then $\dim M^- = n - p$ and U is called a $(p, n-p)$ involution. It is clear that, if a linear transformation commutes with U, it must leave the spaces M^+ and M^- invariant and that there exist invertible linear transformations anticommuting with U if, and only if, $\dim M^+ = \dim M^-$ since

such transformation must take M^+ into M^- and M^- into M^+. Now suppose that T is a similitude which commutes with U, the restriction T_1 (T_2) of T to M^+ (M^-) is a similitude of this subspace with respect to the restriction Q_{M^+} (Q_{M^-}) of Q to M^+ (M^-). Hence, such T will be represented by $T_1 \times T_2$, where T_1 and T_2 have the same ratio. Conversely, if $T_1 \in S(Q_{M^+})$ and $T_2 \in S(Q_{M^-})$ have the same ratio ρ, we can compose both of them to obtain a similitude $T_1 \times T_2 = T$ of $M = M^+ \oplus M^-$ by taking, for any $x = x_1 + x_2$, $x_1 \in M^+$, $x_2 \in M^-$, $Tx = T_1x_1 + T_2x_2$. Then T is a similitude of ratio ρ with respect to Q and it is proper if T_1 and T_2 are both proper or both improper, otherwise T is improper.

The coset of $PS(Q)$ defined by a similitude T will be denoted by $\bar{T}$. It is clear that $\bar{T}_1\bar{T}_2 = \bar{T}_2\bar{T}_1$ implies $T_1T_2 = T_2T_1$ or $T_1T_2 = -T_2T_1$.

2. Let $\Omega(Q)$ be the commutator group of the orthogonal group $O(Q)$. It is well-known that if $\dim M \geqq 3$, the centralizers of $\Omega(Q)$ in $\Omega(Q)$, $O^+(Q)$ and $O(Q)$ consist either of the identity transformation 1_L or of $\pm 1_L$ (see [1, Th. 3.23]). In what follows we will need to know the centralizers of $P\Omega(Q)$, $PO^+(Q)$ and $PO(Q)$ in $PS(Q)$ or $PS^+(Q)$. Now, if we establish that the centralizer of $P\Omega(Q)$ in $PS(Q)$ consist of the coset of the identity, it will follow that the same is true for the centralizers of $PO^+(Q)$ and $PO(Q)$. The essence of the proof is Artin's proof of the theorem just quoted.

LEMMA 1. *Let $\Omega(Q)$ be the commutator group of the orthogonal group $O(Q)$, Q a non-degenerated quadratic form over the vector space M of dimension greater than 2. Then the centralizer in $PS(Q)$ of the group of coset defined by $\Omega(Q)$ consists of the coset of the identity.*

Proof. We consider two different cases.

Case 1. The base field K contains more than 5 elements. In this case, given a 2 dimensional vector space over K with a non-degenerate quadratic form Q_1 the group $\Omega(Q_1)$ contains elements which are not involutions (see e. g. [13]). Now, if N is a non-isotropic two dimensional subspace of M with respect to Q, the commutator group $\Omega(Q)$ contains the group $\Omega(Q_N) \times 1_{N^\perp}$, where Q_N is the restriction of Q to N and $1_{N^\perp}$ is the identity transformation of the vector space $N^\perp$. Let G' be an element of $\Omega(Q_N)$ which is not an involution, then $G = G' \times 1_{N^\perp} \in \Omega(Q)$. If T is a similitude whose coset $\bar{T}$ belongs to the centralizer of $P\Omega(Q)$ in $PS(Q)$ $\bar{G}\bar{T} = \bar{T}\bar{G}$, and we must have $GT = -TG$ or $TG = GT$. We show first that $-TG = GT$ is impossible, for then, if $x \in N^\perp$,

$$xT = xGT = -xTG, \text{ that is, } (xT)G = -xT,$$

but no element of M is taken into its negative by G. Hence $TG = GT$, which implies that T takes $N^{\perp}$ into itself and, consequently, T takes N into itself. Since any one dimensional space of M can be expressed as the intersection of two non-isotropic subspaces of dimension 2, T leaves invariant all the one dimensional subspaces and, therefore, T is a scalar multiplication. In other words T belongs to the coset of the identity.

Case II. Q has index greater than zero (this is always the case if K is a finite field when $\dim M \geqq 3$). If x is any isotropic vector, let N be a non-isotropic 3 dimensional subspace which contains x and let $y \in N$ be a non-isotropic vector orthogonal to x, then there is a rotation G_α of N which leaves x invariant and takes y into $y + \alpha x$ (see [1, p. 133]). The rotation G_α, $\alpha \neq 0$, being the square of $G_{\alpha/2}$ belongs to the commutator group $\Omega(Q_N)$ and hence to $G_\alpha \times 1_{N^\perp} \in \Omega(Q)$. Now, if there were a vector $z \in N$ such that $xG_\alpha = -z$, then $zG_{2\alpha} = zG_\alpha^2 = z$; hence, $G_{2\alpha}$ being a rotation of a 3 dimensional space which leaves invariant the plane spanned by x and z, should be the identity mapping (see [1, Th. 3.17]) but this is impossible, because $yG_{2\alpha} = y + 2\alpha x$. So, if an element $\bar{T} \in PS(Q)$ commutes with the coset of $T_\alpha = G_\alpha \times 1_{N^\perp}$, we must have $TT_\alpha = T_\alpha T$; therefore, T must leave invariant the subspace N_1 spanned by x and $N^\perp$, because N_1 is the subspace of vectors invariant under T_α. Since Kx is the subspace of N_1 orthogonal to N_1, T must take Kx into itself. Hence T leaves invariant every isotropic one dimensional subspace and, consequently, it is a scalar multiple of the identity (see [1, Th. 3.18]).

LEMMA 2. *Let M be a vector space with a non-degenerate quadratic form Q and $\dim M > 4$. If ϕ is an automorphism of the group $S(Q)$ (or $S^+(Q)$) which takes $O^+(Q)$ into itself, ϕ can be written in the form*

$$\phi(T) = V^{-1}TV\chi(T)$$

where $\chi(T)$ is a representation of $S(Q)$ ($S^+(Q)$) into the multiplicative group K' and V is a semi-similitude of Q.

Proof. By the result quoted in Section 1 as (ii) the restriction of ϕ to $O^+(Q)$ can be written in the form

$$\phi(G) = V^{-1}GV\chi(G)$$

Let σ be the automorphism of $S(Q)$ ($S^+(Q)$) defined by

$$\sigma(T) = VTV^{-1}.$$

Then the restriction of the automorphism $\sigma\phi$ to $O^+(Q)$ gives

$$\sigma\phi(G) = G\chi(G) \tag{1}$$

Let T be any similitude (proper similitude) and write $\sigma\phi(T) = TX_T$. Now, if G is any rotation so is $T^{-1}GT$, hence, by (1)

$$\sigma\phi(T^{-1}GT) = T^{-1}GT\chi(T^{-1}GT) \tag{2}$$

On the other hand,

$$\sigma\phi(T^{-1}GT) = \sigma\phi(T^{-1})\sigma\phi(G)\sigma\phi(T) = X_T^{-1}T^{-1}GTX_T\,\chi(G). \tag{3}$$

Comparing (2) and (3) we deduce that X_T commutes or anticommutes with the elements $T^{-1}GT$ for all $G \in O^+(Q)$; therefore, since $T^{-1}GT$ runs over $O^+(Q)$, the coset $\bar{X}_T$ of X_T in $PS(Q)$ ($PS^+(Q)$) belongs to the centralizer of $PO^+(Q)$, that is, X_T is a scalar multiplication which will be called $\chi(T)$. So $\sigma\phi(T) = T\chi(T)$ and $\phi(T) = V^{-1}TV\chi(T)$.

COROLLARY 1. *If $\dim M \geqq 4$, any automorphism ϕ of $S(Q)$ which takes $O(Q)$ into itself has the form described in the lemma.*

Proof. It is enough to substitute $O(Q)$ for $O^+(Q)$ in the proof of the lemma and to use (i) instead of (ii).

COROLLARY 2. *Let $\dim M \geqq 4$. Any automorphism ϕ of $PS(Q)$ which takes $PO(Q)$ into itself is induced by an automorphism of $S(Q)$, that is,*

$$\phi(\bar{T}) = \overline{V^{-1}TV}, \text{ where } V \text{ is a semi-similitude of } Q.$$

The proof is the same as for the lemma with the obvious modifications, namely, $\chi(T)$ does not appear and instead of (ii) we use (iii).

COROLLARY 3. *Let $\dim M = 2m$, $m > 2$ but $m \neq 4$. Any automorphism of $PS(Q)$ (or $PS^+(Q)$) which takes $PO^+(Q)$ into itself is induced by an automorphism of $S(Q)$.*

The proof uses (iv) instead of (iii) and by using (v) we obtain the next corollary.

COROLLARY 4. *Let $\dim M = 2m \geqq 6$ and $\neq 8$. If K is a finite field, every automorphism of $PS(Q)$ (or $PS^+(Q)$) which takes $P\Omega(Q)$ into itself is induced by an automorphism of $S(Q)$.*

3. We are ready now to find the automorphisms of $S(Q)$ and $S^+(Q)$.

THEOREM 1. *Let M be a vector space over the field K and $\dim M \geqq 4$ and let Q be a non-degenerate quadratic form on M. Every automorphism ϕ of the group of similitudes $S(Q)$ is of the form*

$$\phi(T) = V^{-1}TV\chi(T)$$

where V is a semi-similitude of Q and $\chi(T)$ is a representation of $S(Q)$ in K'.

Proof. On account of Corollary 1 of Lemma 2 we only need prove that every automorphism of $S(Q)$ takes $O(Q)$ into itself. Since the orthogonal group is generated by symmetries, that is, $(n-1,1)$ involutions, it is enough to show that every symmetry goes into an orthogonal transformation. If X is a symmetry, $(\phi(X))^2 = 1_L$; therefore, if $\phi(X)$ is not an orthogonal involution, $\phi(X)$ should be a similitude of ratio -1, so that $\phi(X)$ is a similitude of the type studied in [11]. In the proof of Proposition 4 of [11], it is shown that there exist orthogonal transformations anticommuting with the similitudes of such type. Now, if $Y\phi(X) = -\phi(X)Y$, it follows that

$$\phi^{-1}(Y)X = -X\phi^{-1}(Y)$$

which gives a contradiction, because, when $\dim M > 2$, there are no similitudes which anticommute with the symmetries.

THEOREM 2. *Let M be a vector space over the field K, $\dim M = 2m > 4$ and Q be a non-degenerate quadratic form over M. Every automorphism ϕ of the group $S^+(Q)$ is of the form*

$$\phi(T) = V^{-1}TV\chi(T),$$

where V is a semi-similitude and $\chi(T)$ is a representation of $S^+(Q)$ in K'.

Proof. It suffices to prove that under ϕ $O^+(Q)$ goes into itself. Since the group $O^+(Q)$ is generated by the $(n-2,2)$ involutions, if $O^+(Q)$ is not taken into itself by ϕ, at least one, say X, of the $(n-2,2)$ involutions must be taken by ϕ into an involution of $S^+(Q)$ which does not belong to $O^+(Q)$. As before we get that $\phi(X)$ has ratio -1 and is of the type studied in [11]. Now, if $\dim M = 4r$, we get a contradiction since, by [11, Proposition 4], there exist proper similitudes which anticommute with $\phi(X)$ whereas there are no similitudes anticommuting with the $(n-2,2)$ involutions when $\dim M > 4$. If $\dim M = 4r + 2$, this argument does not apply because $S^+(Q)$ does not contain similitudes which anticommute with $\phi(X)$, but we know that in this case (see [11, Prop. 5]) the centralizer of the centralizer of $\phi(X)$ in $S^+(Q)$ consists of similitudes of the form $\alpha_L + \beta_L\phi(X)$ where $\alpha^2 - \beta^2 \neq 0$. If K has more than 3 elements, there exist elements $\alpha \neq 0$ and $\beta \neq 0$ such that $\alpha^2 - \beta^2 \neq 0$. On the other hand, when K has more than 3 elements, the centralizer of the centralizer of a $(n-2,2)$ involution X consists of the elements α_L and $\alpha_L X$, where $\alpha \in K'$. This implies that the centralizer of the centralizer of X is mapped by ϕ into a proper subgroup of the centralizer of the centralizer of $\phi(X)$, which is a contradiction. Therefore $\phi(X) \in O^+(Q)$ and we can apply Corollary 3 of Lemma 2.

We have not taken care yet of the case when K has only 3 elements and $\dim M = 4r + 2$. Notice first that if there do not exist in $S^+(Q)$ involutions of ratio -1, any involution of $S^+(Q)$ is a rotation. On the other hand, if there are involutions of ratio -1, [10, Th. 5 and Corollary 3 of Prop. 4] imply that, when K is the prime field with 3 elements, the group $O^+(Q)$ is generated by the squares of proper similitudes. So that, in any case, any automorphism of $S^+(Q)$ must take $O^+(Q)$ into itself and we can use Corollary 3 of Lemma 2.

4. We assume in this section that K is a finite field and $\dim M > 4$. It is well-known that if K is a finite field and Q a quadratic form on a vector space M over K of dimension greater than 2, then Q has index greater than zero. It is also well-known that, when Q has index greater than zero and $\dim M \geqq 5$, $P\Omega(Q)$ is a simple group (cf. [4, Th. 2] and [6, p. 58]).

Let $(PS(Q))'$ and $(PS^+(Q))'$ be the first commutator groups of $PS(Q)$ and $PS^+(Q)$, respectively. Then

$$PO^+(Q) \supset (PS(Q))' \supset P\Omega(Q)$$

and

$$PO^+(Q) \supset (PS^+(Q))' \supset P\Omega(Q).$$

Now, because of our assumption, it follows that the second commutator groups of $PS(Q)$ and $PS^+(Q)$ are equal to $P\Omega(Q)$. Therefore any automorphism of $PS(Q)$ or $PS^+(Q)$ must take $P\Omega(Q)$ into itself and we can apply Corollary 4 of Lemma 2 to establish

THEOREM 3(1). *Let M be a vector space over a finite field K and $\dim M = 2m \geqq 6$, but $\dim M \neq 8$. Let Q be a non-degenerate quadratic form on M. Any automorphism of $PS(Q)$ or $PS^+(Q)$ is induced by an automorphism of $S(Q)$.*

5. In order to prove the assertion of Theorem 3(1) without the assumption that K is a finite field it will suffice to show that under any automorphism ϕ of $PS(Q)$ or $PS^+(Q)$ the cosets of $(n-2,2)$ involutions go into cosets defined by elements of the rotation group; for, this implies that $PO^+(Q)$ goes into itself and hence Corollary 3 of Lemma 2 can be applied. It will be shown in this section that, in fact, when K has more than 5 elements and $\dim M \geqq 5$ but $\neq 8$, any automorphism of $PS(Q)$ or $PS^+(Q)$ takes the cosets of $(n-2,2)$ involutions into cosets of $(n-2,2)$ involutions.

It will be convenient to describe the different kinds of involutions in $PS(Q)$. Let $T \in S(Q)$, then $\bar{T}$ is an involution in $PS(Q)$ if and only if $T^2 = \alpha_L$; such similitudes will be called projective involutions. The projective involutions of $PS(Q)$ can be divided in 3 classes:

1) T is a scalar multiple of an orthogonal involution, that is, $T = U\beta_L$, where U is an orthogonal involution. Then T has ratio β^2 and $\alpha = \beta^2$. We will say that $\bar{T}$ is the coset of an orthogonal involution since it contains two such involutions, namely, U and $-U$. If either U or $-U$ is a $(n-2,2)$ orthogonal involution we call $\bar{T}$ a $(n-2,2)$ coset.

2) T is a similitude of ratio ρ and $T^2 = -\rho_L$. Such similitudes will be called P-involutions. It was shown in [11, Lemma 1] that the P-involutions are always proper. When $\dim M = 4r+2$, the similitudes anticommuting with a P-involution are improper (see [11, Prop. 4]), so that, in particular, two P-involutions cannot anticommute.

3) T is a similitude of ratio ρ, $T^2 = \rho_L$, but ρ is not a square in K. When $\dim M \geqq 6$ and K has more than 3 elements, no automorphism of $PS(Q)$ or $PS^+(Q)$ takes a $(n-2,2)$ coset into the coset of such a T (see [12, Th. 5]; there is an omission in the statements of this theorem and of Lemma 9, for the lemma is only true if K has more than 3 elements and, therefore Theorem 5 only applies under this assumption).

In the rest of this section it is assumed that K contains more than 5 elements.

First of all we are going to characterize the $(n-2,2)$ cosets among the cosets of orthogonal involutions and state some properties of the P-involutions.

LEMMA 3. *Let M be a vector space with a non-degenerate quadratic form Q and $\dim M > 4$. Let U be an orthogonal involution of $S^+(Q)$ or $S(Q)$. If $(C_{PS^+}(\bar{U}))'$ $((C_{PS}(\bar{U}))')$ is the commutator of the centralizer of $\bar{U}$ in $PS^+(Q)$ $(PS(Q))$, then the center of $(C_{PS^+}(\bar{U}))'$ $((C_{PS}(\bar{U}))')$ contains elements which are not involutions if and only if $\bar{U}$ is a $(n-2,2)$ coset.*

Proof. Let Q^+ and Q^- be the restrictions of Q to the plus and minus spaces of the involution U. Then the centralizer of $\bar{U}$ in $PS^+(Q)$ $(PS(Q))$ contains the cosets of the transformations $T_1 \times T_2$, where T_1 and T_2 are similitudes of Q^+ and Q^-, respectively, with the same ratio and both proper or both improper (without restriction on the parity of T_1 and T_2). When the plus and minus spaces of U have different dimension the centralizer of $\bar{U}$ consists of such cosets only. Therefore, since $\dim M > 4$, when $\bar{U}$ is a $(n-2,2)$ coset the commutator of the centralizer of $\bar{U}$ is contained in the group of cosets determined by $O^+(Q^+) \times O^+(Q^-)$ and contains the cosets defined by $\Omega(Q^+) \times \Omega(Q^-)$. Hence, if U is a $(n-2,2)$ orthogonal involution, the center of $(C_{PS^+}(\bar{U}))'$ $((C_{PS}(\bar{U}))')$ contains an element which is not an involution, namely, $\overline{1_L \times G}$, where $G \in \Omega(Q^-)$ and $G^2 \neq 1_L$.

On the other hand if U is an $(n-p,p)$ orthogonal involution and $n-p \neq 2$, $p \neq 2$, the commutator of the centralizer of $\bar{U}$ in $PS^+(Q)$ or $PS(Q)$ is always contained in the group of cosets defined by $O(Q^+) \times O(Q^-)$ and contains the cosets defined by $\Omega(Q^+) \times \Omega(Q^-)$. Hence the center of $(C_{PS^+}(\bar{U}))'$ or $(C_{PS}(\bar{U}))'$ consists of the coset of the identity or of such coset and the coset defined by $(-1_L) \times 1_L$.

From this lemma and [12, Th. 5], quoted above, we get the following corollary.

COROLLARY. *If $\dim M > 4$, under an automorphism of $PS(Q)$ or $PS^+(Q)$ a $(n-2,2)$ coset goes into a $(n-2,2)$ coset or into the coset of a P-involution.*

LEMMA 4. *Let T be a P-involution. Then any two $(n-2,2)$ orthogonal involutions commuting with T and commuting with each other have their minus spaces orthogonal to each other.*

Proof. It is well-known that two different $(n-2,2)$ orthogonal involutions which commute must have their minus spaces orthogonal to each other or the intersection of these minus spaces is a non-isotropic one dimensional space. If we were in the latter case, since both involutions commute with T, T would leave invariant this one dimensional space. This is impossible, because, if ρ is the ratio of T, $(x, xT) = \rho^{-1}(xT, xT^2) = -(xT, x)$ implies $(x, xT) = 0$, that is, T takes a non-isotropic vector into another vector orthogonal to it.

COROLLARY. *Let $\dim M = 2m > 4$. Let $\{U_i\}$ be a set of $(n-2,2)$ orthogonal involutions which commute with T and with each other, then there are at most m elements in the set $\{U_i\}$.*

We are going to assume that $\dim M = 2m > 4$ and that there exists an automorphism ϕ of $PS(Q)$ or $PS^+(Q)$ which takes a $(n-2,2)$ coset $\bar{U}$ into the coset of a P-involution T. Let U be the $(n-2,2)$ orthogonal involution of $\bar{U}$, let x_1, x_2 be an orthogonal basis of its minus space and $x_3, x_4, \cdots, x_{2m}$ an orthogonal basis of its plus space. Let us define U_i, $i = 2, 3, \cdots, 2m$, as the $(2m-2,2)$ orthogonal involution whose minus space is spanned by the vectors x_1 and x_i. Then $U = U_2$, the $2m-1$ involutions U_i commute with each other and have the property that the product of any two is also a $(n-2,2)$ orthogonal involution. Now, by the Corollary of Lemma 3, we know that the cosets $\phi(U_i)$, $i = 2, \cdots, 2m$, are $(n-2,2)$ cosets or cosets of P-involutions and the product $\phi(U_i)\phi(U_j)$, $i \neq j$, is also a $(n-2,2)$ coset or the coset of a P-involution.

LEMMA 5. *Let U_i, $i = 2, 3, \cdots, 2m$, $2m > 4$, be the set of $(n-2,2)$ involutions described above and let $\phi(U_i) = \bar{T}_i$. Then*

(1) *either all the T_i, $i = 2, \cdots, 2m$ commute with each other, or*

(2) *any two different T_i anticommute.*

Proof. We can assume that if T_i is not a P-involution, it is a $(n-2,2)$ orthogonal involution. It is clear that if $\dim M = 4r+2$ all the T_i commute with each other (see the properties of the P-involution given at the beginning of this section).

On the other hand, if $\dim M = 4r$, it seems possible that some pairs T_i, T_j commute and some others anticommute; however, we will show that if two different T_i commute all of them must commute. To show this we prove first that if T_i and T_j commute, then one of the cosets $\bar{T}_i$, $\bar{T}_j$ or $\bar{T}_i\bar{T}_j$ is a $(4r-2,2)$ coset. We only need prove that, if neither T_i nor T_j is a $(4r-2,2)$ orthogonal involution, $\overline{T_iT_j}$ is a $(4r-2,2)$ coset. Let ρ_i and ρ_j be the ratios of T_i and T_j, then T_iT_j has ratio $\rho_i\rho_j$ and $(T_iT_j)^2 = \rho_i\rho_j$; hence, T_iT_j is not a P-involution and, consequently, $\overline{T_iT_j}$ must be a $(4r-2,2)$ coset. Now, since every $\bar{T}_k$ commutes with $\bar{T}_i$, $\bar{T}_j$ and $\bar{T}_i\bar{T}_j$, T_k must leave invariant the $4r-2$ and 2 dimensional spaces defined by the $(4r-2,2)$ coset; hence the T_k which are P-involutions induce P-involutions in these spaces and we know that in a space of dimension $4r-2$ two P-involutions cannot anticommute. This proves the lemma and, in particular, if all the T_k anticommute with each other there cannot be $(n-2,2)$ involutions among them.

LEMMA 6. *Let $\dim M = 2m$, $m > 2$ but $m \neq 4$. Then any automorphism ϕ of $PS(Q)$ or $PS^+(Q)$ takes a $(2m-2,2)$ coset into a $(2m-2,2)$ coset.*

Proof. Assume that U is a $(2m-2,2)$ orthogonal involution such that $\phi(\bar{U}) = \bar{T}$, where T is a P-involution. Let the U_i, $i = 2, \cdots, 2m$, be defined as before. We are going to show that, when $m > 2$, alternative (1) leads to contradiction and alternative (2) can only occur when $\dim M = 8$.

Case 1. Let $\bar{T}_i = \phi(\bar{U}_i)$ and assume that all the T_i commute with each other. We have seen that this implies that if $\bar{T}_j$ is not a $(n-2,2)$ coset then $\bar{T}_2\bar{T}_j$ is a $(n-2,2)$ coset. By Lemma 4, at most one of the T_j, say T_3, is an orthogonal involution, consequently $\bar{T}_2\bar{T}_j$, $j = 4, 5, \cdots, 2m$, is a $(2m-2,2)$ coset, so that $T_j = T_2V_j\alpha_{jL}$, where V_j is a $(2m-2,2)$ orthogonal involution which commutes with T_2. As for T_3 either $T_3 = T_2V_3\alpha_{3L}$ or $T_3 = V_3$. Now, the V_j, $j = 3, \cdots, 2m$, commute with each other and with T_2, therefore, by the Corollary of Lemma 4, there are at most m different V_j. Hence $m \geqq 2m-2$, that is, $m \leqq 2$.

Case 2. Any two different T_i anticommute. But, if $\dim M = 2^kp$, p not divisible by 2, the maximal number of anticommuting non-singular linear transformations is $2k+1$ (see e.g. [7, Th. 2]; since in our case $T_i^2 = \alpha_l$, the result can be proved by using the theory of Clifford algebras as in Section 6 below). Since the number of T_i is $2^kp - 1$ we must have $2^kp - 1 \leqq 2k+1$ and if $2^kp > 4$, this is only possible when $k = 3$ and $p = 1$, that is to say, if $\dim M = 8$.

From Lemma 6 and Corollary 3 of Lemma 2 we get

THEOREM 3(II). *Let Q be a non-degenerate quadratic form over a vector space M and $\dim M = 2m > 4$ but $\dim M \neq 8$. If the base field K has more than 5 elements, every automorphism ϕ of $PS(Q)$ or $PS^+(Q)$ is of the form*

$$\phi(\bar{T}) = \overline{VTV^{-1}},$$

where V is a semi-similitude of Q.

6. In this section we are going to study the case $\dim M = 8$ which is left out in Thseorem 3(I) and Theorem 3(II). When we deal with $PS^+(Q)$ there can actually exist automorphisms different from the ones described in these theorems (cf. [2]). We will show first that there are not exceptional automorphism of $PO^+(Q)$ if K is a finite field and then it will be proved that even when there exist exceptional automorphisms of $PS^+(Q)$ they cannot be extended to $PS(Q)$. We recall that in the determination of the automorphisms of $PO^+(Q)$ (cf. [4]), when $\dim M = 8$, the trouble arises, as for $PS^+(Q)$, from the fact that it is possible that a $(n-2,2)$ coset is taken by an automorphism ϕ into the coset of a P-involution (the orthogonal P-involutions are the involutions of the second kind of Dieudonné). Now, if it can be proved that the $(n-2,2)$ cosets of $PO^+(Q)$ go into $(n-2,2)$ cosets, the usual proof can be carried through showing that the automorphisms of $PO^+(Q)$ are induced by automorphisms of $O^+(Q)$.

THEOREM 4. *Let M be a 8-dimensional vector space over K and Q a non-degenerate quadratic form over K. Then, if K is a finite field, every automorphism of $PO^+(Q)$ is induced by an automorphism of $O^+(Q)$.*

Proof. When K is a finite field it contains only two different quadratic classes and any subspace of M of dimension greater than 2 contains isotropic vectors; consequently, if U is a $(n-2,2)$ orthogonal involution such that the restriction Q^- of Q to the minus space of U has a square discriminant, U can be expressed as the product of two $(n-2,2)$ orthogonal involutions V_1 and V_2 which commute with each other and such that the discriminants of the restrictions of Q to the minus spaces of V_1 and V_2 are not squares.

To establish our theorem we only need prove that under any automorphism ϕ of $PO^+(Q)$ a $(n-2,2)$ coset $\bar{U}$ cannot be taken into the coset of a P-involution; for then $\bar{U}$ must be taken into another $(n-2,2)$ coset (see [4, Section 39]). Hence, if there are no orthogonal P-involutions there is nothing to prove. If there exist orthogonal P-involutions, Q has a square discriminant and this implies that the coset of a $(n-2,2)$ involution U belongs to $P\Omega(Q)$, the commutator group of $PO^+(Q)$, if and only if the restriction of Q to the minus space of U has a square discriminant, because the spin-norms of U and $-U$ are equal to the discriminant of Q^- (see e.g. [1, Th. 5.17]).

Now, if T is an orthogonal P-involution, $\bar{T} \in P\Omega(Q)$. (This can be seen by computing the spin-norm of T or directly, for, since there exists an orthonormal basis $x_i, y_i, i=1,2,3,4$ of Q such that $x_iT = y_i$ and $y_iT = -x_i$ (cf. e.g. [11, Prop. 1]) $T = T_1ST_1^{-1}S^{-1}$, where T_1 and S are defined as follows.

$$\begin{array}{lll} x_jT_1 = y_j, & y_jT_1 = -x_j, & \text{for } j=1,2, \text{ and} \\ x_jT_1 = x_j, & y_jT_1 = y_j, & \text{for } j=3,4, \text{ and} \\ x_iS = y_{i+2}, & y_iS = x_{i+2}, & \text{where the indexes } i+2 \end{array}$$

should be computed module 4). So, if the restriction of Q to the minus space of the $(n-2,2)$ involution U does not have a square discriminant, then $\phi(\bar{U})$ must be a $(n-2,2)$ coset. Now, if this restriction has a square discriminant, $U = V_1V_2$, where $V_1, V_2 \notin P\Omega(Q)$ and $V_1V_2 = V_2V_1$; hence $\phi(\bar{U}) = \phi(\bar{V}_1)\phi(\bar{V}_2)$, that is, $\phi(\bar{U})$ is the product of two commuting $(n-2,2)$ cosets and, consequently, it cannot be a P-involution.

Remark. [2, Corollary 2 to Th. 1 and Th. 8] imply that under the conditions of Theorem 4 above, when Q has a square discriminant, $PS^+(Q)$ does have exceptional automorphisms.

We return now to the study of $PS(Q)$ and consider two cases.

Case 1. *K is a finite field.* In this case it is easily seen that any 2-dimensional space over K contains similitudes of any ratio with respect to any quadratic form, hence the same is true of M with respect to Q. Then [10, Cor. 3 of Prop. 4] implies that the first commutator group of $PS(Q)$ is $PO^+(Q)$, and, therefore, every automorphism ϕ of $PS(Q)$ induces an automorphism in $PO^+(Q)$. Now, Theorem 4 shows that the proof of Lemma 2 can be applied to the present case to obtain $\phi(\bar{T}) = \overline{V^{-1}TV}$.

Case 2. *K has more than 5 elements.* Let us suppose that under an automorphism ϕ of $PS(Q)$ a $(n-2,2)$ coset $\bar{U}$ is not taken into a $(n-2,2)$ coset. Then, by the corollary of Lemma 3, $\phi(\bar{U}) = \bar{T}$, where T is a P-involution. Consider the set of $(n-2,2)$ orthogonal involutions U_i, $i=2,3,\cdots,8$ defined in the preceding section; the proof of Lemma 6 shows that $\phi(\bar{U}_i) = \bar{T}_i$, where the T_i are P-involutions and any two of them anticommute. Let V be the symmetry whose minus space is spanned by the vector x_1, then $U_iV = VU_i$ and, consequently, if $\phi(\bar{V}) = \bar{W}$, either $WT_i = T_iW$ or $WT_i = -T_iW$, in other words the linear transformation W commutes with some of the T_i and anticommute with the others. Let A_j, $j=2,3,\cdots,8$ be 8×8 matrices representing the linear transformations T_j with respect to a certain basis of M, then $A_j^2 = -\rho_jI$ where I is the 8×8 identity matrix. This means that if N is a 6-dimensional vector space over K and Q' is a quadratic form on N such that there exists an orthogonal basis v_i, $i=1,2,\cdots,6$, with $Q'(v_i) = -\rho_{i+1}$, there exists a homomorphism σ of the Clifford algebra $C(Q')$ defined by Q' into the enveloping algebra of the matrices A_j, $j=2,3,\cdots,7$, namely, $\sigma(v_i) = A_{i+1}$, $i=1,2,\cdots,6$. Since the Clifford algebra $C(Q')$ is simple and $\dim C(Q') = 2^6 = 8\cdot 8$, σ is an isomorphism of $C(Q')$ onto the algebra of 8×8 matrices with entries in K. Now, the only elements of $C(Q')$ which commute with some of the v_i and anticommute with the others are the elements of the Clifford group of the form $\alpha v_{i_1}v_{i_2}\cdots v_{i_h}$; hence, the only linear transformations which commute with some of the T_i and anticommute with the others are of the form $\alpha_1T_{i_1}T_{i_2}\cdots T_{i_h}$. But, it is obvious that $\phi(\bar{V})$ cannot be $\bar{T}_{i_1}\bar{T}_{i_2}\cdots\bar{T}_{i_h}$, because $\bar{V} \neq \bar{U}_{i_1}\bar{U}_{i_2}\cdots\bar{U}_{i_h}$. Therefore any automorphism ϕ of $PS(Q)$ takes the $(6,2)$ cosets into $(6,2)$ cosets and must be of the form $\phi(\bar{T}) = \overline{V^{-1}TV}$.

We sum up these results together with the results of Theorems 3(I) and (II) in our last theorem.

THEOREM 5. *Let M be a vector space with a non-degenerate quadratic form Q and* $\dim M = 2r$, $r > 2$. *Every automorphism of $PS(Q)$, or $PS^+(Q)$ if $\dim M \neq 8$, is induced by an automorphism of $S(Q)$. When $\dim M = 8$, but there do not exist P-involutions, any automorphism of $PS^+(Q)$ is induced by an automorphism of $S^+(Q)$.*

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