THE SPIN REPRESENTATION OF THE UNITARY GROUP

por

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(Continuación)

CHAPTER III

In this chapter the algebra $D(f)$ will be used to obtain representations of the projective group of unitarian similitudes of f into orthogonal groups. The spaces of the representations are the subspaces of $D(f)$ of degree $2i$, $i = 1, 2, \ldots, n-1$, which are vector spaces over K. We map the unitarian similitude S into the linear transformation induced in the subspace of $D(f)$ of degree $2i$ by the automorphism of $C^{+}(Q)$ associated to S. It will be seen that in this form we get a representation of the projective group of unitarian similitudes.

In § 1 we define a symmetric bilinear form on the subspaces of degree $2i$ of $D(f)$, $i = 1, 2, \ldots, n-1$. Then it will be shown that for any i this form is non-degenerate and the linear transformation induced in one of these subspaces by the automorphism associated to a unitarian similitude is an orthogonal transformation with respect to this symmetric bilinear form.

As before it will be assumed that the hermitian form f is non-degenerate and the characteristic of F is zero or greater than (M : F).

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§ 1

We define a symmetric bilinear form (x, y) on the subspace of degree r of $C(Q)$ in the following way:

Let a and b be any two elements of the subpace of degree r of $C(Q)$. Then we take as value of (a, b) the component of degree zero of $a\,b^*$. It is obvious that the form so defined is bilinear. Since a and b are homogeneous elements, the anti-automorphism $*$ leaves them invariant if $r = 4m$ or $r = 4m + 1$ and takes them into their opposites if $r = 4m + 2$ or $r = 4m + 3$. Therefore, since the component of degree zero is left invariant by the homogeneous anti-automorphism $*$, $a\,b^*$ has the same component of degree zero that $(a\,b^*)^* = b\,a^*$. Then $(a, b) = (b, a)$, which shows that (x, y) is a symmetric form.

LEMMA 1. The symmetric bilinear form defined on the spaces of degree r of $C(Q)$ is non-degenerate if and only if Q is non-degenerate.

PROOF. Let $z_1, z_2, \ldots, z_N$ be an orthogonal basis with respect to Q, of the vector space on which Q is defined. Then the elements

$$z_{i_1} z_{i_2} \ldots z_{i_r}, \qquad i_1 < i_2 < \ldots < i_r$$

form an orthogonal basis of the space of degree r with respect to the form (x, y). If all the vectors z_i, $i = 1, 2, \ldots, N$ are non-isotropic and

$$a = z_{i_1} z_{i_2} \ldots z_{i_r}, \qquad (a, a) = Q(z_{i_1})\, Q(z_{i_2}) \ldots Q(z_{i_r}) \neq 0$$

and the form (x, y) is non-degenerate. But, if one of the vectors z_i is isotropic, there are isotropic vectors in the chosen orthogonal basis of the space of degree r.

The form (x, y) induces a symmeric bilinear form on the subspaces of degree $2i$ of $D(f)$. It will be proved that this induced bilinear form is non degenerate.

We are going to use the notation of chapter II; in particular u_i, v_i, r_i, s_i have the same meaning that in chapter II, § 1, where x_i, $y_i = \theta x_i$, $i = 1, 2, \ldots, n$ is an orthogonal basis with respect to Q.

Lemma 2. Let

$$a = w_{i_1} w_{i_2} \ldots w_{i_m} \quad \text{and} \quad b = w'_{i_1} w'_{i_2} \ldots w'_{i_m},$$

where w_{i_k}, w'_{i_k} stand for u_{i_k} or v_{i_k}. Then

$$(a, b) = (w_{i_1}, w'_{i_1})(w_{i_2}, w'_{i_2}) \ldots (w_{i_m}, w'_{i_m})$$

and is different from zero if and only if $a = b$, that is,

$$w_{i_k} = w'_{i_k}, \quad k = 1, 2, \ldots, m.$$

Proof. The elements w_h and w_f, $h \neq f$, commute with each other and $w_h^* = -w_h$ for any h. Therefore

$$ab^* = w_{i_1} w'^*_{i_1} w_{i_2} w'^*_{i_2} \ldots w_{i_m} w'^*_{i_m}. \tag{1}$$

The product $w_j w_j'^*$ has one of the following forms,

$$\left.\begin{aligned}
v_j v_j^* &= -v_j^2 = 2\,Q(x_{2j-1})\,Q(x_{2j}) + 2\rho^{-1} x_{2j-1} x_{2j} y_{2j-1} y_{2j},\\
u_j u_j^* &= -u_j^2 = -2\rho\,Q(x_{2j-1})\,Q(x_{2j}) -\\
&\qquad -2\,x_{2j-1} y_{2j-1} x_{2j} y_{2j} = -\rho\, v_j v_j^*\\
u_j v_j^* &= -2\,Q(x_{2j-1})\, x_{2j} y_{2j} + 2\,Q(x_{2j})\, x_{2j-1} y_{2j-1}\\
v_j u_j^* &= -u_j v_j^*.
\end{aligned}\right\} \tag{2}$$

The subindices $2i_h - 1$, $2i_h$ of x and y in the product $w_{i_h} w'^*_{i_h}$ are different of the subindices $2i_k - 1$, $2i_k$ of x and y in $w_{i_k} w'^*_{i_k}$, if $h \neq k$. If the index systems of two elements of the form

$$x_1^{\varepsilon_1} y_1^{\delta_1} \ldots x_n^{\varepsilon_n} y_n^{\delta_n}$$

have no common indices the degree of their product is the sum of the degrees of these elements. Therefore the zero component of

$$\prod_{j=1}^{m} w_{i_j} w'^*_{i_j}$$

is the product of the zero components of each factor $w_{i_j} w'^*_{i_j}$. Equating the zero components of (1) we have

$$(a, b) = (w_{i_1}, w'_{i_1})\,(w_{i_2}, w'_{i_2}) \ldots (w_{i_m}, w'_{i_m}).$$

On the other hand the equalities (2) show that

$$(w_{i_j}, w'_{i_j}) \begin{cases} = 0 & \text{if} \quad w_{i_j} \neq w'_{i_j} \\ \neq 0 & \text{if} \quad w_{i_j} = w'_{i_j} \end{cases}$$

and therefore

$$(a, b) \begin{cases} = 0 & \text{if} \quad a \neq b \\ \neq 0 & \text{if} \quad a = b \end{cases}.$$

Lemma 2'. If in lemma 2 we assume that w_{i_k} and w'_{i_k} stand for s_{i_k} or r_{i_k} the lemma is also true.

The proof is the same as before, but instead of (2) we have

$$\left.\begin{aligned}
s_j s_j^* &= -s_j^2 = 2\,Q(x_{2j-1})\,Q(x_{2j}) + 2\rho^{-1} x_{2j-1} y_{2j-1} x_{2j} y_{2j}\\
r_j r_j^* &= -r_j^2 = -2\rho\,Q(x_{2j-1})\,Q(x_{2j}) -\\
&\qquad -2\,x_{2j-1} y_{2j-1} x_{2j} y_{2j} = -\rho\, s_j s_j^*\\
s_j r_j^* &= 2\,Q(x_{2j-1})\, x_{2j} y_{2j} + 2\,Q(x_{2j})\, x_{2j-1} y_{2j-1}\\
r_j s_j^* &= -s_j r_j^*
\end{aligned}\right\} \tag{2'}$$

In the proof of lemma 6 of chapter II we have defined by induction on h a basis of the space of $2h$ different indices. To simplify the notation we had supposed there that the indices were $1, 2, \ldots, 2h$ and had pointed out how to deduce a basis for any space of $2h$ different indices from the basis of the space of indices $1, 2, \ldots, 2h$. The bases so defined for the index spaces of any system of $2h$ different indices will be called the canonical bases.

We are going to recall the form of these bases and, at the same time, we introduce a new notation which will be used later on. Let us assume that we know already the form of the elements of the canonical basis of a space of an index system formed by

$2j$ different indices, $2j \leqslant h$. Let $i_1 < i_2 < \ldots < i_{2j}$ be $2j$ numbers chosen among the numbers $1, 2, \ldots, h$. The $\binom{2j}{j}$ elements $c_1, c_2, \ldots, c_{\binom{2j}{j}}$ of the canonical basis of the space of indices $i_1, i_2, \ldots, i_{2j}$ can be written as follows,

$$c_f = \sum_{\nu_i=0}^{\nu_i=1} \alpha^f_{\nu_1 \ldots \nu_{2j}} p_{i_1}^{\nu_1} p_{i_2}^{\nu_2} \ldots p_{i_{2j}}^{\nu_{2j}} \quad \text{where} \quad \alpha^f_{\nu_1 \ldots \nu_{2j}} \in K, \tag{3}$$

$$p_{i_k}^{\nu_k} = \begin{cases} x_{i_k} & \text{if} \quad \nu_k = 0 \\ y_{i_k} & \text{if} \quad \nu_k = 1 \end{cases}$$

and the sum extends over the 2^{2j} different j-tuples deduce from $(\nu_1, \nu_2, \ldots, \nu_{2j})$ letting $\nu_i = 0, 1$.

If we change the meaning of $p_{i_k}^{\nu_k}$ taking

$$p_{i_k}^{\nu_k} = \begin{cases} s_{i_k} & \text{if} \quad \nu_k = 0 \\ r_{i_k} & \text{if} \quad \nu_k = 1 \end{cases} \tag{4}$$

we get a canonical element of the space of indices $2i_1 - 1$, $2i_1, \ldots, 2i_{2j} - 1$, $2i_{2j}$. This element will be denoted by C_f.

Let $i'_1 < i'_2 < \ldots < i'_{h-2j}$ be the complementary set of $i_1, i_2, \ldots, i_{2j}$ with respect to $1, 2, \ldots, h$. Let us mulpiply each C_f by each one of the elements D_g, $g = 1, 2, \ldots, 2^{h-2j}$ deduce from $w_{i'_1} w_{i'_2} \ldots w_{i'_{h-2j}}$ sustituting $u_{i'_k}$ or $v_{i'_k}$ for each $w_{i'_k}$. The elements $C_f D_g$ form the canonical basis for the subspace of the index family $i_1, i_2, \ldots, i_{2j}$, which is a subspace of the space of indices $1, 2, \ldots, 2h$. The union of the canonical bases of all the subspaces belonging to the different index families of the space of indices $1, 2, \ldots, 2h$ is the canonical basis for this index space.

It will be proved first that if the spaces of degree less than $2h$ have an orthogonal basis of non-isotropic vectors, the space of degree $2h$ has an orthogonal basis with the same property. This conclusion will be reached through a sequence of lemmas.

Lemma 3. The subspaces of the same degree defined by two different index systems are orthogonal to each other.

Proof. The elements a and b of degree $2h$ are linear combinations of elements

$$x_1^{\varepsilon_1} y_1^{\delta_1} x_2^{\varepsilon_2} y_2^{\delta_2} \ldots x_n^{\varepsilon_n} y_n^{\delta_n}, \quad \varepsilon_i, \delta_i = 0,1; \quad \text{and} \quad \sum_i \varepsilon_i + \sum_i \delta_i = 2h. \tag{5}$$

The product of two elements of the form (5) is a homogeneous element and it has degree zero if and only if both elements are equal. If a and b belong to two index subspaces with different index systems the elements (5) which appear in the expression of a do not appear in the expression of b. Therefore the component of degree zero of the product $a\,b$ is zero. Since

$$a\,b^* = (-1)^{\binom{2h}{2}} a\,b, \qquad (a, b) = 0.$$

It follows from this lemma that if we have an orthogonal basis of non-isotropic vectors for each one of the index subspaces of degree $2h$, the union of these bases is an orthogonal basis of non-isotropic vectors for the space of degree $2h$.

Lemma 4. If the index spaces of degree $2(h-1)$ have orthogonal bases of non-isotropic vectors the same is true for the subspaces of an index system of degree $2h$ in which at least one index appears twice.

Proof. Let k be an index which appears twice in a given index system of degree $2h$. We consider the index system of degree $2(h-1)$ deduced from the given system of degree $2h$ by leaving out the pair of indices $k\,k$. If we multiply each element of any basis of the subspace of the index system of degree $2(h-1)$ by $x_k y_k$ we get a basis for the subspace of the given index system. Let us denote by m_j, $j = 1, 2, \ldots, N$ the elements of an orthogonal basis of non-isotropic vectors of the index subspace of degre $2(h-1)$, i. e., $(m_i, m_j) = 0$ if $i \neq j$, $(m_i, m_i) \neq 0$. Since

$$m_i x_k y_k (m_j x_k y_k)^* = m_i x_k y_k y_k x_k m_j^* = -\rho\, Q\,(x_k)^2\, m_i m_j^*,$$

equating the components of degree zero of the right hand and left hand expressions we get

$$(m_i x_k y_k, m_j x_k y_k) = -\rho Q(x_k)^2 (m_i, m_j) \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

which proves the lemma.

Now we need to find orthogonal bases of non-isotropic vectors for the index subspaces of any system of $2h$ different indices. Without loss of generality we can assume that we are dealing with the subspace of indices $1, 2, \dots, 2h$. We are going to consider this subspace as the direct sum of the subspaces of the different index families (see chapter II, proof of lemma 6).

Lemma 5. The subspaces of the different index families of the space of index $1, 2, \dots, 2h$ are orthogonal to each other.

Proof. Let a and b be two elements of the canonical basis of the space of indices $1, 2, \dots, 2h$, belonging to two different subspaces of the index families. Let k be an index which belongs to the family of indices of the subspace containing b and does not belong to the family of indices of the subspace containing a.

Let us express a and b as linear combination of elements of the form

$$t_1 t_2 \dots t_h \tag{6}$$

where t_j stands for u_j, v_j, r_j or s_j, $j = 1, \dots, h$. Then, in the terms which appear in the expression of b, t_k stands for s_k or r_k, whereas in the terms which appear in the expression of a, t_k stands for u_k or v_k.

On the other hand the antiautomorphism * takes the element $\alpha t_1 t_2 \dots t_h$ into $(-1)^h \alpha t_1 t_2 \dots t_h$ for t_i and t_j commute if $i \neq j$ and $t_j^* = -t_j$, since t_j has degree 2.

Then $a b^* = (-1)^h a b$ is a sum of products and each one of these products contains one of the following pairs of factors, u_k, r_k; u_k, s_k; v_k, r_k or v_k, s_k among other factors wich commute with these ones. Since

$$u_k r_k = u_k s_k = v_k r_k = v_k s_k = 0 \qquad a b^* = (-1)^h a b = 0$$

and therefore $(a, b) = 0$.

Lemma 6. If the canonical bases of the spaces of $2j$ different indices, $2j \leqslant h$, are orthogonal bases of non-isotropic vectors, the canonical basis of a subspace of any index family belonging to the index system $1, \dots, 2h$ is an orthogonal basis of non-isotropic vectors.

Proof. It has been seen that the elements of the canonical basis of the subspace of the index family $i_1, i_2, \dots, i_{2j}$ have the form $C_f D_g$.

It will be first proved that any two elements of the canonical basis are orthogonal to each other. We consider two different cases

Case I. Let us suppose that the two elements are $E_1 = C_{f_1} D_{g_1}$ and $E_2 = C_{f_2} D_{g_2}$ where $g_1 \neq g_2$. This means that if $i'_1, \dots, i'_{h-2j}$ is the complementary set of $i_1, i_2, \dots, i_{2j}$ the factor D_g which has the form $w_{i'_1} w_{i'_2} \dots w_{i'_{h-2j}}$ is different for the two given elements.

In the expression of the elements of the subspace of an index family the r_k or s_k which appear in the factor C_f have different indices that the u_m or v_m which appear in D_g. Therefore any C_f commutes with any D_g. Then,

$$E_1 E_2^* = C_{f_1} D_{g_1} (C_{f_2} D_{g_2})^* = C_{f_1} D_{g_1} D_{g_2}^* C_{f_2}^* = C_{f_1} C_{f_2}^* D_{g_1} D_{g_1}^*.$$

The product $C_{f_1} C_{f_2}$ is a linear combination of elements (5) where only the indices $2i_1 - 1, 2i_1, \dots, 2i_{2j} - 1, 2i_{2j}$ may appear. None of these indices appears in the product of $D_{g_1} D_{g_1}^*$; therefore the component of degree zero of $E_1 E_2^*$ is the product of the components of degree zero of $C_{f_1} C_{f_2}^*$ and $D_{g_1} D_{g_2}^*$, that is,

$$(E_1, E_2) = (C_{f_1}, C_{f_2})(D_{g_1}, D_{g_2}).$$

Now lemma 2 asserts that

$$(D_{g_1}, D_{g_2}) \begin{cases} = 0 & \text{if } g_1 \neq g_2 \\ \neq 0 & \text{if } g_1 = g_2 \end{cases}$$

Therefore $(E_1, E_2) = 0$ if $g_1 \neq g_2$.

CASE II. $E_1 = C_{f_1} D_g$, $E_2 = C_{f_2} D_g$. Since we suppose that $E_1 \neq E_2$ we must have $f_1 \neq f_2$.

The C_f are defined by (3) where the $p_{i_k}^{\nu_k}$ have the meaning given by (4) and if we take

$$p_{i_k}^{\nu_k} = \begin{cases} x_{i_k} & \text{if} \quad \nu_k = 0 \\ y_{i_k} & \text{if} \quad \nu_k = 1 \end{cases}$$

we get the element c_f of the canonical basis of the space of indices $i_1, i_2, \ldots, i_{2j}$. Since we assume that the canonical basis of such space is an orthogonal basis of non-isotropic vectors, the component of degree zero of $c_{f_1} c_{f_2}^*$ must be zero if $f_1 \neq f_2$ and different from zero if $f_1 = f_2$. In the product

$$c_{f_1} c_{f_2}^* = \left(\sum_{\nu_i = 0}^{1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} p_{i_1}^{\nu_1} \ldots p_{i_{2j}}^{\nu_{2j}} \right) \left(\sum_{\nu_i = 0}^{1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} p_{i_{2j}}^{\nu_{2j}} \ldots p_{i_1}^{\nu_1} \right)$$

we only get elements of degree zero when we multiply two terms with the same set of values for $\nu_1, \nu_2, \ldots, \nu_{2j}$. Therefore

$$\left. \begin{aligned} (c_{f_1}, c_{f_2}) &= \left(\sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} (-\rho)^{\sum_i \nu_i} \right) \times \\ &\times Q(x_{i_1}) \ldots Q(x_{i_{2j}}) \begin{cases} = 0 & \text{if} \quad f_1 \neq f_2 \\ \neq 0 & \text{if} \quad f_1 = f_2 \end{cases} \end{aligned} \right\} \tag{7}$$

If we compute the component of degree zero of the product $C_{f_1} C_{f_2}^*$ taking into account lemma 2' and the equalities (2') we get

$$\left. \begin{aligned} (C_{f_1}, C_{f_2}) &= \sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} \left(p_{i_1}^{\nu_1}, p_{i_1}^{\nu_1} \right) \ldots \left(p_{i_{2j}}^{\nu_{2j}}, p_{i_{2j}}^{\nu_{2j}} \right) = \\ &= \left(\sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} (-\rho)^{\sum_i \nu_i} \right) 2^{2j} Q(x_{2i_1 - 1}) \\ Q(x_{2i_1}) \ldots Q(x_{2i_{2j}-1}) \, Q(x_{2i_{2j}}) &\begin{cases} = 0 & \text{if} \quad f_1 \neq f_2 \\ \neq 0 & \text{if} \quad f_1 = f_2 \end{cases} \end{aligned} \right\} \tag{8}$$

for $\prod_{k=1}^{2j} Q(x_{2i_k - 1}) \, Q(x_{2i_k}) \neq 0$, and (7) shows that

$$\sum \alpha_{\nu_1 \ldots \nu_{2j}}^{f_1} \alpha_{\nu_1 \ldots \nu_{2j}}^{f_2} (-\rho)^{\sum_i \nu_i} \begin{cases} = 0 & \text{if} \quad f_1 \neq f_2 \\ \neq 0 & \text{if} \quad f_1 = f_2 \end{cases}.$$

Therefore it has been proved that the vectors $C_f D_g$ are orthogonal to each other. It remains to be proved that they are non-isotropic. This is immediate, for if

$$E = C_f D_g, \qquad (E, E) = (C_f, C_f)(D_g, D_g);$$

since (8) proves that $(C_f, C_f) \neq 0$ and lemma 2 asserts that

$$(D_g, D_g) \neq 0, \qquad (E, E) \neq 0.$$

Lemmas 5 and 6 prove that if the canonical basis of any space of a system of $2j$ different indices, $2j \leqslant h$, is an orthogonal basis of non-isotropic vectors, the same is true for the canonical basis of a space of a system of $2h$ different indices. If $j = 1$, u_1 and v_1 form the canonical basis for the space of indices 1, 2 and from this basis we get by induction and substitution of indices the canonical bases of all the spaces of $2h$ different indices. The equalities (2) show that u_1 and v_1 form a canonical basis of non-isotropic vectors. Therefore the canonical basis of any index space of a system of $2h$ different indices is an orthogonal basis of non-isotropic elements.

If we determine now orthogonal bases of non-isotropic vectors for each space of an index system of degree $2h$ in which at least an index appears twice, lemma 3 asserts that the union of these bases and the canonical bases of the spaces of index systems of $2h$ different indices form an orthogonal basis of non-isotropic vectors for the space of $D(f)$ of degree $2h$. Using now lemma 4, it suffices to prove that the space of degree 2 has a basis with this property. Since the element $x_i y_i$ is non-isotropic, for

$$(x_i y_i, x_i y_i) = -\rho \, Q(x_i) \neq 0,$$

the elements

$$x_i y_i, \quad u_{ij}, \quad v_{ij}, \quad i, j = 1, 2, \ldots, n, \quad i < j,$$

form an orthogonal basis of non-isotropic elements for the space of degree 2. Therefore we have established

THEOREM 1. Let Q be the quadratic form associated to the non-degenerate hermitian form f. Then the symmetric bilinear forms (a, b) that has been defined on the spaces of degree r of $C(Q)$ induce non-degenerate symmetric bilinear forms on the spaces of degree $2i$ of $D(f)$, $i = 1, \ldots, n$.

§ 2

In chapter I we have associated to any unitarian similitude of f an automorphism of $C^+(Q)$, where Q is the quadratic form associated to f. If the invertible element $c \in C(Q)$ defines an inner automorphism inducing in $C^+(Q)$ the automorphism associated to a unitarian similitude S, then theorem 3 of chapter II asserts that $c \in D(f)$. Therefore the automorphism of $C^+(Q)$ associated to a unitarian similitude induces in $D(f)$ an inner automorphism. Moreover it is known that such automorphism is homogeneous of degree 0 and therefore induces linear transformations in the spaces of $D(f)$ of degree $2i$, $i = 0, 1, \ldots, n$.

THEOREM 2. The linear transformation of the space of degree $2h$ of $D(f)$, $h = 0, 1, \ldots, n$, induced by an automorphism of $C^+(Q)$ associated to a unitarian similitude is an orthogonal transformation with respect to the symmetric bilinear form (a, b).

PROOF. Let a and b be two elements of degree $2h$ of $D(f)$. By definition (a, b) is the component of degree zero of $a b^*$.

Let σ be the automorphism associated to the unitarian similitude S. Since σ is homogeneous of degree o, it commutes with the antiautomorphism *. Moreover σ leaves K invariant elementwise; therefore the zero component of $(a b^*)^\sigma = a^\sigma (b^\sigma)^*$ coincides with the zero component of $a b^*$, which implies $(a, b) = (a^\sigma, b^\sigma)$. Hence the linear transformation induced by σ in the space of degree $2h$ of $D(f)$ is an orthogonal transformation.

The quadratic form associated to the bilinear form (a, b) defined on the space D_{2h} of $D(f)$ of degree $2h$ will be denoted by Q_{2h}.

The mapping φ_{2h} which takes the element U of the group of unitarian similitudes of f, $S(f)$, into the orthogonal transformation induced in D_{2h} by the automorphism of $C^+(Q)$ associated to U is a homomorphism of the group $S(f)$ into the orthogonal group $O(Q_{2h})$. Moreover, since $D(f)$ has been defined as the subalgebra of $C^+(Q)$ consisting of the elements invariant under the automorphisms of $C^+(Q)$ associated to the homotecies of f, φ_{2h} maps these homotecies into the identity of $O(Q_{2h})$. Therefore by means of φ_{2h} we get a homomorphism ψ_{2h} of the factor group of $S(f)$ by the group of homotecies into the orthogonal group $O(Q_{2h})$. Since the factor group of $S(f)$ by the group of homotecies, which is its center, is the projective group of unitarian similitudes $PS(f)$, we get.

THEOREM 3. For $h = 1, 2, \ldots, n-1$, ψ_{2h} defines a representation of $PS(f)$ into the orthogonal group of the space D_{2h} with respect to the form (a, b).

It is not difficult to prove now that when K has more than 5 elements each one of these representations is faithful. We omit here this proof since we will publish a more refined result in a paper where the irreducible components of each one of these representations will be determined.

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CRONICA

PROFESOR PEÑA SERRANO

El 30 de octubre último falleció en Madrid el ilustre Profesor de la Escuela de Montes, D. Fernando Peña Serrano.

Ingeniero de Montes, y con una vocación extraordinaria a la docencia, pasó pronto a desempeñar la cátedra de Cálculo y Mecánica Racional en la Escuela Técnica del Cuerpo. Su competencia, sus desvelos por todos los problemas que afectaron a su especialidad o a la Escuela, le hacían mostrarse siempre en un estado de preocupación constante.

Quizá una de sus últimas actuaciones fué la intervención en un Tribunal de Oposiciones para cubrir una cátedra de Matemáticas de dicha Escuela.

Socio antiguo de la R. Sociedad Matemática Española, formó parte de la Junta directiva, en la que laboró siempre con entusiasmo.

Escribió varios trabajos sobre distintos temas, y entre ellos están los que publicó en nuestra Revista, «Sobre la traslación paralela infinitesimal» y «Un método para determinar los niveles de energía del oscilador armónico».

Descanse en paz el querido amigo, que supo cristianamente llevar con paciencia una dolencia crónica, de la que no mostró queja alguna, y siempre amable y cariñoso con todos, supo, con la bondad en los labios, llevar con humildad e hidalguía, ese don, privilegiado de los elegidos por Dios, hasta el último momento de su vida.

G. R.