la segunda de las cuales muestra que el radio de curvatura solamente depende de la distancia del punto al plano tangente a los axoides ($x = 0$).

La ecuación del plano osculador a la trayectoria de M es:

$$(y_M - y)\frac{dh}{dt} + (z - z_M)\, x_M \frac{dr}{dt} = 0 \qquad [37]$$

que muestra que todos son paralelos al eje de las x.

Las ecuaciones del eje de curvatura son:

$$\left. \begin{aligned} x = x_M; \quad & -(x - x_M)\frac{dr}{dt} y_M + (y - y_M)\, x_M \frac{dr}{dt} + \\ & + \frac{dh}{dt}(z - z_M) - V^2 = 0 \end{aligned} \right\} \qquad [38]$$

que muestra que todos son paralelos al plano $x = 0$. La intersección del eje de curvatura con el plano $z = z_M$, tiene por coordenadas:

$$x = x_M; \quad z = z_M; \quad y = y_M + \frac{V^2}{x_M \dfrac{dr}{dt}} \qquad [39]$$

que comparada con las fórmulas del movimiento plano [28], muestran que el eje de curvatura pasa por el centro de curvatura de la trayectoria de la proyección del móvil sobre el plano $z = z_M$.

En este caso no existe nunca ningún punto de aceleración nula, ni de inflexión, ya que el radio de curvatura por [36] nunca puede ser nulo.

IRREDUCIBLE REPRESENTATIONS OF THE PROJECTIVE GROUP OF UNITARIAN SIMILITUDES

por

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RESUMEN

En este articulo se obtienen representaciones fieles e irreducibles del grupo proyectivo de semejanzas unitarias en grupos ortogonales. Las restricciones de estas representaciones al grupo projectivo unitario o a su subgrupo engendrado por las simetrías de la forma hermitiana siguen siendo irreducibles. Las conclusiones obtenidas sólo son válidas si la caracteristica del cuerpo base es cero o mayor que la dimensión del espacio sobre el cual está definida la forma hermitiana. En el caso de que esta dimensión sea menor que 5 es necesario suponer que el cuerpo base posee más de 25 elementos.

Este trabajo es continuación de [1], por lo que se hacen referencias constantes a las definiciones y resultados allí incluidos.

This paper is a sequel of [1]. The irreducible representations of the projective group of unitarian similitudes which we find here are the irreducible components of the representations defined in the last chapter of the mentioned paper. We will recall briefly how we had arrived to such representations.

Let f be a hermitian form on the finite dimensional vector space M over the commutative field F of characteristic different from 2. Let J be the involutive automorphism of F associated to f. We suppose that J is different from the identity and denote by K the subfield of elements of F invariant under J. Then $F = K(\theta)$, where $\theta = -\theta$, $\theta^2 = \rho \in K$.

Then M being a vector space over F has an underlying structure of vector space over K. If we consider M as a vector space over K we can define on it a symmetric bilinear form in the following way,

$$(x, y) = f(x, y) + f(y, x).$$

Taking $Q(x) = \frac{1}{2}(x, x)$, Q is a quadratic form which will be referred to as the quadratic form associated to f; we assume that f is non-degenerate, in which case Q is also non-degenerate.

Any unitarian similitude of f is a similitude relative to Q of the same ratio. Therefore if C (Q) denotes the Clifford algebra defined by Q, there exist automorphisms of the subalgebra $C^+(Q)$ associated to the unitarian similitudes of f, and these automorphisms are homogeneous with respect to the gradation of $C^+(Q)$ as a vector space (see [1], Ch. I, § 1). Let D (f) be the subalgebra of $C^+(Q)$ consisting of the elements invariant under the automorphisms associated to the homotecies of f; the automorphisms of $C^+(Q)$ associated to the unitarian similitudes induce linear transformations on the subspaces of degree $2h$, $h = 1, 2, \ldots, n$, of D (f). When the caracteristic of F is zero or greater than $n = (M : F)$ the algebra D (f) has been studied in [1], where it has been defined also a non-degenerate quadratic form Q_{2h} on the subspace of degree $2h$ of D (f) (Ch. III, § 1) and it was proved that the linear transformations induced in this subspace by the automorphisms associated to the unitarian similitudes define a representation of the projective group of unitarian similitudes in the orthogonal group determine by Q_{2h} (Ch. III, § 2). These representations will be study in this paper and their irreducibles components will be determined. That is, we will find the decomposition of the subspace of D (f) of degree $2h$ into a direct sum of subspaces irreducible under the automorphisms induced in D (f) by the elements of the projective group of unitarian similitudes. With the exception of the trivial representation, all the others irreducible representations thus obtained are faithful.

Throughout this paper it will be assumed that the characteristic of F is zero or greater than (M : F) and that, in any case, K has more than 5 elements.

§ 1

In order to study the representation of the projective group of unitarian similitudes into the orthogonal group O (Q_{2h}), we are going to use the decomposition of the subspace of D (f) of degree $2h$ into a direct sum of index subspaces defined in [1] Ch. II, § 1. First of all we recall some facts which have been proved there.

Let $x_1, x_2, \ldots, x_n$ be an orthogonal basis of f and x_i, $y_i = \theta x_i$, $i = 1, 2, \ldots, n$, the corresponding orthogonal basis of Q. Any element of D (f) of degree $2h$ can be expressed as a linear combination of elements of the form

$$x_1^{\varepsilon_1} y_1^{\delta_1} x_2^{\varepsilon_2} y_2^{\delta_2} \ldots x_n^{\varepsilon_n} y_n^{\delta_n}$$

where

$$\varepsilon_i, \delta_i = 0, 1, \qquad \sum_i (\varepsilon_i + \delta_i) = 2h.$$

The index system of the element

$$\alpha\, x_1^{\varepsilon_1} y_1^{\delta_1} \ldots x_n^{\varepsilon_n} y_n^{\delta_n},$$

$\alpha \in K$, was defined as the set of subindices of the x's and y's with exponent equal to 1. The subspace of D (f) whose elements are sums of elements with the same index system is called the index subspace of the given index system. The algebra D (f) is the direct sum of its index subspaces.

LEMMA 1.—The product of

$$\sum_{j=1}^{2h} Q(x_{i_j})^{-1} x_{i_j} y_{i_j}$$

by any element of D (f) belonging to the subspace of indices $i_1 < i_2 < \ldots < i_{2h}$ is zero.

PROOF.—To simplify the notation we make $p_i = Q(x_i)^{-1} x_i y_i$ and without loss of generality we can assume that the indices are $1, 2, \ldots, 2h$.

The element p_i commutes with x_k and y_k, if $k \neq i$, and it anticommutes with x_i and y_i. Therefore p_i anticommutes with any element c of the subspace of indices $1, 2, \ldots, 2h$ if $i \leqslant 2h$ and commutes with c if $i > 2h$. Then $\sum_{i=1}^{2h} p_i$ anticommutes with c and $\sum_{i=2h+1}^{n} p_i$ commutes with it.

On the other hand $r_1 = \sum_{i=1}^{n} p_i$ belongs to the center of $D(f)$ (see [1] Ch. I, lemma 2). Therefore

$$\sum_{i=1}^{2h} p_i = \sum_{j=1}^{n} p_j - \sum_{k>2h}^{n} p_k$$

must commute with c. Since the characteristic of K is different from 2, this implies that

$$\left(\sum_{i=1}^{2h} p_i\right) c = 0 .$$

It was seen in [1], Ch. II, § 2, that any element of $C^+(Q)$, and in particular any element of $D(f)$, of degree $2h$, could be written in a unique way as a linear combination of products of the form

$$t_1 t_2 \ldots t_h \tag{1}$$

where t_i, $i = 1, 2, \ldots, h$, stands for any of the 4 following terms

$$u_i = u_{2i-1,\,2i} = x_{2i-1} y_{2i} - y_{2i-1} x_{2i},$$
$$v_i = v_{2i-1,\,2i} = x_{2i-1} x_{2i} - \rho^{-1} y_{2i-1} y_{2i},$$
$$r_i = r_{2i-1,\,2i} = x_{2i-1} y_{2i} + y_{2i-1} x_{2i} \quad \text{or}$$
$$s_i = s_{2i-1,\,2i} = x_{2i-1} x_{2i} + \rho^{-1} y_{2i-1} y_{2i}.$$

LEMMA 2.—Let $d \neq 0$ be any element of $D(f)$ of the subspace of indices $1, 2, \ldots, 2h$. There exists a reordination of the indices $1, 2, \ldots, 2h$ such that in the expression of d as linear combination of elements of the form (1) there is an element with non zero coefficient where t_1 stands for u_1 or v_1.

PROOF.—Let us assume that $d \neq 0$ is a linear combination of elements (1) where t_1 always stands for r_1 or s_1. Since

$$(p_1 - p_j) r_{1j} = (p_1 - p_j) s_{1j} = 0$$

we must have $(p_1 - p_2) d = 0$. If we interchange the indices 2 and $j > 2$, and for the new order

$$\{1', 2', \ldots, (2h)'\} = \{1, j, 3, \ldots, (j-1), 2, (j-1), \ldots, 2h\}$$

in the expression of d as linear combination of elements of the elements of the form (1) $t_{1'}$ always stands for $r_{1'} = r_{1j}$ or $s_{1'} = s_{1j}$, then $(p_1 - p_j) d = 0$. Hence, if this is true for any $j \geqslant 2$,

$$\sum_{j=2}^{2h} (p_1 - p_j) d = 2h\, p_1 d - \sum_{k=1}^{2h} p_k d = 0$$

and taking into account lemma 1 we get $2h\, p_1 d = 0$, which is a contradiction for $2h\, p_1$ has inverse.

We are going to study now the automorphisms of $D(f)$ associated to the elements of the group $U^{\pm}(f)$ generated by the symmetries of f. It is known that the automorphism associated to the symmetry of f which leaves elementwise invariant the hyperplane orthogonal, with respect to f, to the non-isotropic vector x and takes x into $-x$, is the homogeneous inner automorphism defined by the element $c = x y$, where $y = \theta x$ (cf. [1], Ch. II, theorem 3).

We consider the following cases,

I) $x = x_i$. Then $c = x_i y_i$ commutes with x_j and y_j if $j \neq i$, and anticommutes with x_j and y_j. Therefore the inner automorphism defined by c leaves invariant u_{jk} and v_{jk} if $j, k \neq i$, and q_j for every j. On the other hand

$$c^{-1} u_{jk} c = - u_{jk}, \quad c^{-1} v_{jk} c = - v_{jk}, \quad \text{if} \quad j = i \quad \text{or} \quad k = i.$$

In general, if U_i is the automorphis associated to the unitarian symmetry which leaves elementwise invariant the hyperplane orthogonal to x_i, U_i leaves invariant the elements of the index subspaces whose index systems do not contain i or where i appears twice and takes an element of an index subspace whose index system contains i only once in its opposite.

II) $x = x_i + \alpha x_j$. Since x is non isotropic

$$\mu = Q(x_i + \alpha x_j) = Q(x_i) + \alpha^2 Q(x_j) \neq 0,$$

but we will assume also that

$$\nu = Q(x_i) - \alpha^2 Q(x_j) \neq 0.$$

It is obvious that $\alpha \neq 0$ can always be chosen as to fulfil these conditions because we have assumed that K has more than 5 elements.

Then $c = (x_i + \alpha x_j)(y_i + \alpha y_j)$ commutes with x_m and y_m if $m \neq i, j$; anticommutes with v_{ij} and s_{ij}, and

$$c^{-1} x_i c = \mu^{-1} (y_i + \alpha y_j)^{-1} (x_i + \alpha x_j) x_i (x_i + \alpha x_j)(y_i + \alpha y_j) =$$

$$= \mu^{-1} (y_i + \alpha y_j)^{-1} \left(\nu x_i + 2 \alpha Q(x_i) x_j \right) (y_i + \alpha y_j) =$$

$$= - \mu^{-1} \left(\nu x_i + 2 \alpha Q(x_i) x_j \right)$$

$$c^{-1} y_i c = - \mu^{-1} \left(\nu y_i + 2 \alpha Q(x_i) y_j \right)$$

$$c^{-1} x_j c = - \mu^{-1} \left(- \nu x_j + 2 \alpha Q(x_j) x_i \right)$$

$$c^{-1} y_j c = - \mu^{-1} \left(- \nu y_j + 2 \alpha Q(x_j) y_i \right).$$

Therefore

$$c^{-1} u_{km} c = u_{km}; \quad c^{-1} v_{km} c = v_{km}, \quad \text{if} \quad k, m \neq i, j$$

$$c^{-1} v_{ij} c = - v_{ij}, \quad c^{-1} s_{ij} c = - s_{ij};$$

$$c^{-1} p_i c = \mu^{-2} \left(\nu^2 p_i + 4 \alpha^2 Q(x_i) Q(x_j) p_j + 2 \alpha \nu u_{ij} \right)$$

$$c^{-1} p_j c = \mu^{-2} \left(4 \alpha^2 Q(x_i) Q(x_j) p_i + \nu^2 p_j - 2 \alpha \nu u_{ij} \right),$$

$$c^{-1} (p_i - p_j) c = \mu^{-2} \left((\nu^2 - 4 \alpha^2 Q(x_i) Q(x_j)) (p_i - p_j) + 4 \alpha \nu u_{ij} \right);$$

$$c^{-1} u_{ij} c = \mu^{-2} \left(4 \alpha Q(x_i) Q(x_j) \nu (p_i - p_j) + (4 \alpha^2 Q(x_i) Q(x_j) - \right.$$

$$\left. - \nu^2) u_{ij} \right);$$

$$c^{-1} r_{ij} c = - r_{ij};$$

$$c^{-1} u_{ik} c = - \mu^{-1} \left(\nu u_{ik} + 2 \alpha Q(x_i) u_{jk} \right), \quad \text{if} \quad k \neq j;$$

$$c^{-1} u_{jk} c = - \mu^{-1} \left(- \nu u_{jk} + 2 \alpha Q(x_j) u_{ik} \right), \quad \text{if} \quad k \neq i,$$

III) $x = x_i + \alpha y_j$ and

$$\mu = Q(x_i) - \alpha^2 \rho Q(x_j) \neq 0, \qquad \nu = Q(x_i) + \rho \alpha^2 Q(x_j) \neq 0.$$

Then

$$c = (x_i + \alpha y_j)(y_i + \alpha \rho x_j)$$

commutes with x_m and y_m if $m \neq i, j$; anticommutes with u_{ij} and r_{ij}; and

$$c^{-1} x_i c = - \mu^{-1} \left(\nu x_i + 2 \alpha Q(x_i) y_j \right),$$

$$c^{-1} y_i c = - \mu^{-1} \left(\nu y_i + 2 \rho \alpha Q(x_i) x_j \right),$$

$$c^{-1} x_j c = \mu^{-1} \left(\nu x_j + 2 \alpha Q(x_j) y_i \right),$$

$$c^{-1} y_j c = \mu^{-1} \left(\nu y_j + 2 \rho \alpha Q(x_j) x_i \right).$$

Therefore

$$c^{-1} u_{km} c = u_{km}, \quad c^{-1} v_{km} c = v_{km}, \quad \text{if} \quad k, m \neq i, j;$$
$$c^{-1} u_{ij} c = - u_{ij}, \quad c^{-1} r_{ij} c = - r_{ij};$$

$$c^{-1}(p_i - p_j) c = \mu^{-2}\left((\nu^2 + 4\alpha^2 \rho\, Q(x_i) Q(x_j)(p_i - p_j) + 4\alpha\rho\nu v_{ij}\right)$$
$$c^{-1} v_{ij} c = - \mu^{-2}\Big((\nu^2 + 4\alpha^2 \rho\, Q(x_i) Q(x_j)) v_{ij} +$$
$$+ 4\alpha\, Q(x_i) Q(x_j) \nu (p_i - p_j)\Big)$$
$$c^{-1} s_{ij} c = - s_{ij}.$$

Given an element b of degree $2h$, we want to find some properties of the smallest subspace $V(b)$ of the space of degree $2h$ which is invariant under the linear transformations defined by the homogeneous automorphisms of $D(f)$ associated to the unitarian transformations of $U^{\pm}(f)$. It is clear that the elements of $V(b)$ are linear combinations of b and its images under the mentioned automorphisms, and that $V(b)$ contains the subspace $V(d)$ defined by any $d \in V(b)$.

LEMMA 3.—If b is an element of $D(f)$ of degree $2h$, the sum of its projection on the index subspaces whose index systems contain i only once (do not contain the index i or contain it twice) belongs to $V(b)$.

PROOF.—The element

$$\frac{1}{2}(b - c^{-1} b c) \quad \left(\frac{1}{2}(b + c^{-1} b c)\right),$$

where $c = x_i y_i$, belongs to $V(b)$ and has the form stated in the lemma.

LEMMA 4.—If the element $b \in D(f)$ of degree $2h$ has a projection different from zero on the subspace of indices

$$i_1 < i_2 < \ldots < i_{2h},$$

such projection belongs to $V(b)$.

PROOF.—The element

$$b' = \frac{1}{2}\left(b - (x_{i_1} y_{i_1})^{-1} b\, x_{i_1} y_{i_1}\right) \in V(b),$$

is the sum of the proyections on the index subspaces whose index systems contain i_1 only once. Repeating this process for each index i_j, we get the elements of $V(b)$.

$$b'' = \frac{1}{2}\left(b' - (x_{i_2} y_{i_2})^{-1} b'\, x_{i_2} y_{i_2}\right), \ldots, =$$
$$b^{(2h)} = \frac{1}{2}\left(b^{(2h-1)} - (x_{i_{2h}} y_{i_{2h}})^{-1} b^{(2h-1)} x_{i_{2h}} y_{i_{2h}}\right),$$

and $b^{(2h)}$ is the projection of b on the subspace of indices $i_1, \ldots, i_{2h}$.

LEMMA 5.—Let $b = a\, z_{ij}\, a' \in D(f)$ be an element of degree $2h$, where a and a' are sum of elements belonging to index subspaces whose index systems contain neither i nor j, and z_{ij} stands for either u_{ij}, or v_{ij}, or $p_i - p_j$. Then the three elements $a\, u_{ij}\, a'$, $a\, v_{ij}\, a'$, $a\,(p_i - p_j)\, a'$ belong to $V(b)$.

PROOF. — It will be sufficient to prove the three following statements,

1) if $b = a\, u_{ij}\, a'$, then $a\,(p_i - p_j)\, a' \in V(b)$,
2) if $b = a\,(p_i - p_j)\, a'$, then $a\, u_{ij}\, a'$, $a\, v_{ij}\, a' \in V(b)$,
3) if $b = a\, v_{ij}\, a'$, then $a\,(p_i - p_j)\, a' \in V(b)$.

1) If $b = a\, u_{ij}\, a'$, we take the symmetry of case II defined by

$$c = (x_i + \alpha x_j)(y_i + \alpha y_j)$$

and because of the assumption on a and a' we have

$$c^{-1} b c = c^{-1} a\, u_{ij}\, a' c = a\, c^{-1} u_{ij} c\, a' =$$
$$= \mu^{-2} a \left(4\alpha\, Q(x_i) Q(x_j) \nu (p_i - p_j) + (4\alpha^2 Q(x_i) Q(x_j) - \nu^2) u_{ij}\right) a'.$$

Applying lemma 3 we know that

$$4\,\mu^{-2}\,\alpha\,Q(x_i)\,Q(x_j)\,\nu\,a\,(p_i - p_j)\,a' \in V(b).$$

Therefore, since

$$4\,\mu^{-2}\,\alpha\,Q(x_i)\,Q(x_j)\,\nu \neq 0, \qquad a\,(p_i - p_j)\,a' \in V(b),$$

2) If $b = a\,(p_i - p_j)\,a'$ we take the symmetry of case II (III) defined by

$$c = (x_i + \alpha\,x_j)\,(y_i + \alpha\,y_j)\Big(c = (x_i + \alpha\,y_j)\,(y_i + \rho\,\alpha\,x_j)\Big). \quad \text{Since}$$

$$c^{-1}\,b\,c = a\,c^{-1}\,(p_i - p_j)\,c\,a' =$$

$$= \mu^{-2}\,a\Big((\nu^2 - 4\,\alpha^2\,Q(x_i)\,Q(x_j))\,(p_i - p_j) + 4\,\alpha\,\nu\,u_{ij}\Big)\,a' \in V(b)$$

$$c^{-1}\,b\,c = \mu^{-2}\,a\Big((\nu^2 + 4\,\alpha^2\,\rho\,Q(x_i)\,Q(x_j)\,(p_i - p_j) +$$

$$+ 4\,\alpha\,\rho\,\nu\,v_{ij})\,a' \in V(b)\Big)$$

and

$$4\,\mu^{-2}\,\alpha\,\nu \neq 0,$$

$a\,u_{ij}\,a'$ $(a\,v_{ij}\,a')$ belongs to $V(b)$ by lemma 3.

3) If $b = a\,v_{ij}\,a'$ we take the symmetry of case III defined by

$$c = (x_i + \alpha\,y_j)\,(y_i + \alpha\,\rho\,x_j).$$

Then

$$c^{-1}\,b\,c = -\,\mu^{-2}\,a\Big((\nu^2 + 4\,\alpha^2\,\rho\,Q(x_i)\,Q(x_j))\,v_{ij} +$$

$$+ 4\,\alpha\,Q(x_i)\,Q(x_j)\,\nu\,(p_i - p_j)\Big)\,a' \in V(b),$$

and by lemma 3,

$$a\,(p_i - p_j)\,a' \in V(b).$$

LEMMA 6.—Let $b \in D(f)$ be a linear combination of products of the form

$$t_1\,t_2 \ldots t_h,$$

where t_i stands for

$$u_i = u_{2i-1,\,2i}, \qquad v_i = v_{2i-1,\,2i}, \qquad s_i = s_{2i-1,\,2i}$$

or

$$r_i = r_{2i-1,\,2i}\,;\ i = 1, 2, \ldots, h.$$

Then fixing an index j, $1 \leqslant j \leqslant h$, the element b_1 derived from this expression by substituting 0 for v_j, r_j, s_j (u_j, r_j, s_j), belongs to $V(b)$.

PROOF.—We apply a symmetry of type II (III) with

$$c = (x_{2j-1} + \alpha\,x_{2j})\,(y_{2j-1} + \alpha\,y_{2j}\Big(c = (x_{2j-1} + \alpha\,y_{2j})\,(y_{2j-1} + \alpha\,\rho\,x_{2j})\Big),$$

The element

$$b' = b + c^{-1}\,b\,c \in V(b)$$

is obtained from b by substituting 0 for v_j, r_j, and s_j (u_j, r_j, and s_j), and

$$\mu^{-2}\Big(4\,\alpha\,Q(x_{2j-1})\,Q(x_{2j})\,\nu\,(p_{2j-1} - p_{2j}) + 8\,\alpha^2\,Q(x_{2j-1})\,Q(x_{2j})\,u_{2j-1,\,2j}\Big),$$

$$\Big(-\,\mu^{-2}\,(4\,\alpha\,Q(x_{2i-1})\,Q(x_{2j})\,\nu\,(p_{2j-1} - p_{2j}) +$$

$$+ 8\,\rho\,\alpha^2\,Q(x_{2j-1})\,Q(x_{2j})\,v_{2j-1,\,2j})\Big)$$

for u_j (v_j).

The projection of b' on the subspace of indices $1, 2, \ldots, 2h$ also belongs to $V(b)$ and it is equal to $\beta\,b_1$, where β is a non zero coefficient. Therefore $b_1 \in V(b)$.

LEMMA 7.— Let $b \in D(f)$ be an element of degree $2h$ which has a projection different from zero on a subspace with $2h$ different indices which we can assume to be $1, 2, \ldots, 2h$. Then $V(b)$ contains all the elements of the form

$$z_{j_1 j_2} z_{j_3 j_4} \ldots z_{j_{2k-1} j_{2k}} \tag{2}$$

where $z_{j_{2k-1} j_{2k}}$ stands for $u_{j_{2k-1} j_{2k}}$, $v_{j_{2k-1} j_{2k}}$ or $p_{j_{2k-1}} - p_{j_{2k}}$ and $j_1, j_2, \ldots, j_{2h}$ is any reordenation of $1, 2, \ldots, 2h$.

PROOF.—The projection d of b on the subspace of indices $1, \ldots, 2h$ belongs to $V(b)$. By lemma 2 we know that by a suitable reordenation of the indices in the expression of d as linear combination of elements of the form (1), there is an element with non-zero coefficient where $t_{1'}$ stands for

$$u_{1'} = u_{1'2'} \quad \text{or} \quad v_{1'} = v_{1'2'}.$$

Then by lemma 6 we know that the element d_1 sum of the terms where $t_{1'}$ stands for $u_{1'2'}$ $(v_{1'2'})$ also belongs to $V(b)$. That is, d_1 is equal to $u_{1'2'}$ or $v_{1'2'}$ by a linear combination different from zero of products of the form

$$t_{2'} t_{3'} \ldots t_{h'}$$

and it is easily seen that this linear combination also belongs to $D(f)$. Taking now a reordenation of the indices $3', 4', \ldots, (2h)'$ we can consider that some $t_{2''}$ stands for $u_{2''}$ or $v_{2''}$ and applying again lemma 6 we get an element d_2 which is the product of $w_{1'2'}$ $w_{3''4''}$ by a linear combination of elements

$$t_{3''} t_{4''} \ldots t_{h''}$$

and such linear combination belongs to $D(f)$. Proceeding in this form we get an element of the form

$$w_{i_1 i_2} w_{i_3 i_4} \ldots w_{i_{2k-1} i_{2k}} \in V(b) \tag{3}$$

where each $w_{i_{2k-1} i_{2k}}$ represents one of the elements $u_{i_{2k-1}, i_{2k}}$ or $v_{i_{2k-1}, i_{2k}}$. We simplify the notation by writing now $i_1, i_2, \ldots, i_{2h}$ as $1, 2, \ldots, 2h$.

Then by lemma 5 we know that all the elements of the form

$$z_{1,2} z_{3,4} \ldots z_{2h-1, 2h}$$

where $z_{2k-1, 2k}$ stands for $u_{2k-1, 2k}$, $v_{2k-1, 2k}$ or $p_{2k-1} - p_{2k}$ belong to $V(b)$.

To complete the proof of the lemma we need to show that for any reordenation $j_1, j_2, \ldots, j_{2h}$ of the indices,

$$z_{j_1 j_2} z_{j_3 j_4} \ldots z_{j_{2k-1} j_{2k}} \in V(b). \tag{4}$$

Since it is possible to pass from one ordenation to another one by interchanging at each step two succesive elements, it is sufficient to prove that if $j_1, j_2, \ldots, j_h$ is obtained from $1, 2, \ldots, 2h$ inverting the order of two consecutive elements, (4) belongs to $V(b)$. This is obvious if we go from $1, 2, \ldots, 2k-1, 2k, \ldots, 2h$ to $1, 2, \ldots, 2k, 2k-1, \ldots, 2h$, for

$$z_{2k-1, 2k} = \pm z_{2k, 2k-1}$$

Let us suppose then that we go from

$$1, 2, \ldots, 2k, 2k+1, \ldots, 2h$$

to

$$1, 2, \ldots, 2k+1, 2k, \ldots, 2h.$$

We take the element of $V(b)$.

$$e = u_{1,2} \ldots u_{2k-1, 2k} u_{2k+1, 2k+2} \ldots u_{2h-1, 2h}$$

and apply to it the simmetry of type II defined by

$$c = (x_{2k} + \alpha x_{2k+1})(y_{2k} + \alpha y_{2k+1}).$$

Then

$$c^{-1} e c = \mu^{-2} u_{12} \ldots \left(\nu u_{2k-1, 2k} + 2\alpha Q(x_{2k}) u_{2k-1, 2k+1}\right) \cdot$$
$$\cdot \left(-\nu u_{2k+1, 2k+2} + 2\alpha Q(x_{2k+1}) u_{2k, 2k+2}\right) \ldots u_{2h-1, 2h} \in V(b)$$

and since its projection on the subspace of indices $1, 2, \ldots, 2h$

$$e' = \mu^{-2} u_{12} \ldots \left(-\nu^2 u_{2k-1, 2k} u_{2k+1, 2k+2} + \right.$$
$$\left. + 4 \alpha^2 Q(x_{2k}) Q(x_{2k+1}) u_{2k-1, 2k+1} u_{2k, 2k+2} \right) \ldots u_{2h-1, 2h}$$

belongs to $V(b)$,

$$e' + \mu^{-2} \nu^2 e = 4 \mu^{-2} \alpha^2 Q(x_{2k}) Q(x_{2k+1}) \cdot$$
$$\cdot u_{12} u_{34} \ldots u_{2k-1, 2k+1} u_{2k, 2k+2} \ldots u_{2h-1, 2h} \in V(b).$$

Therefore all the elements

$$z_{12} \ldots z_{2k-1, 2k+1} z_{2k, 2k+2} \ldots z_{2h-1, 2h} \in V(b).$$

Corollary.—If the element b has a non-zero projection on the subspace of indices $1, 2, \ldots, 2h$, then V (b) contains this subspace.

This is a consequence of [1], Ch. II, lemma 6 taking into account that the canonical elements are defined as linear combination of elements of the form (3).

Lemma 8.—If the element $b \in D(f)$ of degree $2h$ has a non-zero projection on one index subspace with $2h$ different indices, then $V(b)$ contains all the index subspaces with $2h$ different indices

Proof.—If there are only $2h$ different indices the lemma is given by the preceding corollary. When there are more than $2h$ different indices, in view of this corollary, it will be sufficient to show that $V(b)$ contains an element of any index subspace with $2h$ different indices. Moreover, since we can pass from an index system A with $2h$ different indices to another index system B of the same type by substituting succesively an index of B not in A for an index of A not in B, it only need be proved that if $V(b)$ contains an element of the subspace with the index system A it also contains an element of the subspace whose index system is obtained from A by such a substitution.

To simplify the notación we can assume that

$$A = \{1, 2, \ldots, 2h\}$$

and

$$B = \{1, 2, \ldots, 2h-1, 2h+1\}.$$

By the corollary

$$d = u_{1, 2} u_{3, 4} \ldots u_{2h-1, 2h} \in V(b)$$

and taking a symmetry of type II where

$$c = (x_{2h} + \alpha x_{2h+1})(y_{2h} + \alpha y_{2h+1}),$$

$c^{-1} d c$ has a non zero projection on the subspace of index system B. Hence this subspace is contained in $V(b)$.

Definition.

$$V^{(h)}, h = 0, 1, \ldots, \left[\frac{n}{2}\right],$$

is the subspace (over K) of elements of degree $2h$ generated by all the elements of the form (4) where $j_1, j_2, \ldots, j_{2h}$ is any set of $2h$ different indices. In particular $V^{(0)} = K$.

Lemmas 7 and 8 show that any subspace of the space of degree $2h$ of $D(f)$ invariant under the automorphims associated to the elements of $U^{\pm}(f)$ and containing an element with a non zero projection on one index subspace with $2h$ different indices must contain $V^{(h)}$.

§ 2

In [1], Ch. III theorem 3 we have established a homomorphism of the group $PS(f)$, the projective group of unitarian similitudes of f, into a group of linear transformations of the subspace D_{2h} of $D(f)$ of degree $2h$, $h = 1, 2, \ldots, n-1$. It was proved also that these linear transformations are orthogonal transformations with respect to the quadratic form Q_{2h} defined there and that this form is non-degenerate. We will say that this group of linear transformations, which are orthogonal relative to Q_{2h}, is defined by $(PS(f), h)$.

Now we are ready to find the irreducibe components of D_{2h} relative to such group of linear transformations. Since these transformations were defined by means of homogeneous inner automorphisms of $D(f)$, the element $r_1 = \sum_{i=1}^{n} p_i$ is invariant under these automorphisms. On the other hand if we take an element e of degree $2(h-1)$, $e r_1$ might have a non zero component of degree $2h$. In fact, let us suppose that e belongs to an index subspace whose index system is

$$A = \{ i_1, i_2, \ldots, i_{2h-2} \}.$$

Then the degree of $e p_i$ is:

a) $2(h-2)$, if A contains the index i twice,

b) $2(h-1)$, if A contains the index i only once,

c) $2h$, if A does not contain the index i. In this case $e p_i \neq 0$ is an element of the subspace of indices $i_1, i_2, \ldots, i_{2h-2}, i, i$.

Therefore, if $2(h-1) < n$, the element $e r_1$ has a non zero component of degree $2h$, for there exists at least an index i which is not contained in A and the projection of

$$e r_1 = e p_i + e \sum_{j \neq i} p_j$$

on the subspace of indices

$$i_1, i_2, \ldots, i_{2h-2}, i, i$$

is $e p_i$; hence the component of degree $2h$ of $e r_1$ is different from zero. Moreover, this component is sum of elements whose index systems contain at least an index which appears twice. This proves

LEMMA 9.—Let e be an element of any index subspace of degree $2(h-1) < n$. Then $e r_1$ has a non zero component of degree $2h$. Moreover, this component has zero projection on the index subspace with $2h$ different indices.

Let us denote by $[a]_h$ the component of degree $2h$ of the element a and by $[W]_h$ the subspace of degree $2h$ consisting of the components of degree $2h$ of the elements of the space W.

LEMMA 10.—The subspace

$$[D_{2(h-1)} r_1]_h \subset D_{2h}$$

is invariant under the group of linear transformations defined by $(PS(f), h)$.

PROOF.—Let U be the linear transformations of D_{2h} defined by the automorphism σ associated to an element of $PS(f)$. If $a \in [D_{2(h-1)} r_1]_h$, then $a = [a' r_1]_h$, where $a' \in D_{2(h-1)}$. Hence, since σ is a homogeneous inner automorphism,

$$a U = a^\sigma = [a' r_1]_h^\sigma = \left[(a' r_1)^\sigma\right]_h = [a'^\sigma r_1]_h \in [D_{2(h-1)} r_1]_h \text{ Q. E. D.}$$

Since all the elements of $[D_{2(h-1)} r_1]_h$ have zero projection on the index subspaces with $2h$ different indices, $[D_{2(h-1)} r_1]_h$ is orthogonal, with respect to Q_{2h}, to these index subspaces (see [1], Ch. III, lemma 3). Moreover, $[D_{2(h-1)} r_1]_h$ is ortogonal to $V^{(h)}$ because, if b is an element of any index subspace of degree $2h$ with $2h$ different indices, b is orthogonal to $[D_{2(h-1)} r_1]_h$, therefore $V(b)$ is also orthogonal to it and $V^{(h)} \subset V(b)$. In order to show now that $V^{(h)}$ is precisely the orthogonal complement of $[D_{2(h-1)} r_1]_h$ we are going to established a preliminary lemma.

LEMMA 11.—Let $f_1, f_2, \ldots, f_n$ be any elements commuting with each other and let

$$X(i_1 i_2 \ldots i_{k-1}; i_k i_{k+1}) = f_{i1} f_{i2} \ldots f_{ik-1} (f_{ik} - f_{ik+1}),$$

where the i_j are all different, $1 \leqslant i_j \leqslant n$ and $k < n$ is fixed. Then any element of the form

$$f_{m1} f_{m2} \ldots f_{mk} - f_{q1} f_{q2} \ldots f_{qk},$$

where the m_j are all different, the q_j are all different and $1 \leqslant m_j \leqslant n$, $1 \leqslant q_j \leqslant n$, is a sum of elements

$$X(i_1 \ldots i_{k-1}; i_k i_{k+1}).$$

PROOF.—We will use induction on the number N of indices $q_1, \ldots, q_k$ not contained in $m_1, \ldots, m_k$; therefore $N \leqslant n-k$, $N \leqslant k$. If N is 1, we just have the

$$X(i_1 \ldots i_{k-1}; i_k i_{k+1}).$$

Let us assume that the lemma holds for $N < h \leqslant n-k$, $h \leqslant k$; that is, by assumption

$$C_{(h-1)} = f_1 f_2 \ldots f_k - f_1 f_2 \ldots f_{k-(h-1)} f_{k+1} f_{k+2} \ldots f_{k+(h-1)} = \\ = \sum X(i_1 \ldots i_{k-1}; i_k i_{k+1}).$$

Then

$$C_{(h)} = f_1 f_2 \ldots f_k - f_1 f_2 \ldots f_{k-h} f_{k+1} f_{+2} \ldots f_{k+h} = \\ = C_{(h-1)} + X\big(1, 2, \ldots, k-h, k+1, k+2 \ldots k+ \\ + (h-1); \quad k-(h-1), k+h\big) \text{ Q. E. D.}$$

LEMMA 12.—For

$$h \leq \left[\frac{n}{2}\right], \qquad D_{2h} = [D_{2(h-1)} r_1]_h \oplus V^{(h)}$$

and the restrictions of Q_{2n} to $[D_{2(h-1)} r_1]_h$ and $V^{(h)}$ are non degenerate.

PROOF.—We will show first that

$$D_{2h} = [D_{2(h-1)} r_1]_h + V^{(h)}.$$

To prove this, it is sufficient to show that the right hand space contains any element of any index subspace of degree $2h$. Since the elements of any index subspace with $2h$ different indices belong to $V^{(h)}$, we only have to care about the elements b whose index systems contains at least an index which appears twice.

To simplify the notation we assume that the index system is

$$1, 1, 2, 2, \ldots, k, k, n-2(h-k)+1, n-2(h-k)+2, \ldots, n$$

and $1 \leqslant k \leqslant h$. Therefore $b = p_1 p_2, \ldots, p_k b'$, where b' is an element of the subspace with index system $n-2(h-k)+1, \ldots, n$ and can be expressed as a linear combination of elements

$$w_{i_1 i_2} w_{i_3 i_4} \ldots w_{i_{2(h-k)-1} i_{2(h-k)}}$$

where $i_1, i_2, \ldots, i_{2(h-k)}$ are different reordenations of $n-2(h-k)+1, \ldots, n$. Hence, if we write $n-2(h-k) = n'$, by lemma 2

$$b' \sum_{i=n'+1}^{n} p_i = 0.$$

Because of this equality it will suffice to show that

$$p_1 p_2 \ldots p_k = \left[c \sum_{i=1}^{n'} p_i\right]_k + d \qquad (5)$$

where c and d satisfy the following conditions:

Condition 1.—c has degree $2(k-1)$ and is a linear combination of elements whose index systems contain only the indices $1, 2, \ldots, n'$;

Condition 2.—$d \in V^{(k)}$ and is a linear combination of elements

$$(p_{j_1} - p_{j_2})(p_{j_3} - p_{j_4}) \ldots (p_{j_{2k-1}} - p_{j_{2k}})$$

where $1 \leqslant j_f \leqslant n'$ and the j_f's are all different.

For then

$$p_1 p_2 \dots p_h b' = \left[c \sum_1^{n'} p_i\right]_h b' + d b' = \left[b' c \sum_1^{n'} p_i\right]_h + d b' =$$

$$= \left[b' c \left(\sum_1^{n'} p_i + \sum_{n'+1}^{n} p_j\right)\right]_h + d b' = [b' c r_1]_h + d b' ,$$

$b' c$ has degree $2(h-1)$ and $d b' \in V^{(h)}$. Let us notice that, since $n' = n - 2h + 2k$ and

$$h \leq \left[\frac{n}{2}\right], \quad 2k \leq n' .$$

If $k = 1$,

$$p_1 = \frac{1}{n'} \sum_1^{n'} p_i + \frac{1}{n'} \sum_{j=2}^{n'} (p_1 - p_j) = c \sum_1^{n'} p_i + d$$

and

$$c = \frac{1}{n'}, \quad d = \frac{1}{n'} \sum_{j=2}^{n'} (p_1 - p_j)$$

satisfy conditions 1 and 2.

This implies in particular that, if we take into account $2k \leqslant n'$, (5) with conditions 1 and 2 is true for $n' = 2$ and $n' = 3$. We assume that this is also true for any $n' < s$ and any k such that $2k \leqslant n'$, and we are going to show that then it is true for $n' = s$ and any k such that $2k \leqslant s$. Since we know that (5) with conditions 1 and 2 is true for $n' = s$ if $k = 1$, we need considerer only the case $k = f + 1 > 1$ and $2f + 2 \leqslant s$. But $2f + 2 \leqslant s$ implies $2f \leqslant s - 2 = s'$ and by assumption

$$p_1 p_2 \dots p_f = \left[c_{1\dots f} \sum_1^{s'} p_i\right]_f + d_{1\dots f}$$

where $c_{1\dots f}$ and $d_{1\dots f}$ satisfy conditions 1 and 2.

Then

$$X(1, 2, \dots, f; s'+1, s'+2) = p_1 p_2 \dots p_f (p_{s'+1} - p_{s'+2}) =$$

$$= \left[c_{1\dots f} \sum_1^{s'} p_i\right]_f (p_{s'+1} - p_{s'+2}) + d_{1\dots f}(p_{s'+1} - p_{s'+2}) =$$

$$= \left[c_{1\dots f}(p_{s'+1} - p_{s'+2}) \sum_1^{s} p_i\right]_{f+1} + d_{1\dots f}(p_{s'+1} - p_{s'+2})$$

and

$$c(1, \dots, f: s'+1, s'+2) = c_{1\dots f}\, p_{s'+1} - p_{s'+2}) ,$$

$$d(1, \dots, f; s'+1, s'+2) = d_{1\dots f}(p_{s'+1} - p_{s'+2})$$

satisfy conditions 1 and 2. By substituting the indices we can get in the same form any

$$X(i_1, \dots, i_f; i_{f+1}, i_{f+2}),\ 1 \leq i_f \leq s' + 2 = s$$

Applying now lemma 11 we know that

$$p_{j_1} \cdots p_{j_{f+1}} - p_{g_1} \cdots p_{g_{f+1}}$$

is a sum of elements

$$X(i_1, i_2, \dots, i_f; i_{f+1}, i_{f+2}) =$$

$$= \left[c(i_1, \dots, i_f: i_{f+1}, i_{f+2}) \sum_1^{s} p_i\right]_{f+1} + d(i_1, \dots, i_f; i_{f+1}, i_{f+2})$$

$1 \leqslant i_q \leqslant s$, and the i_q's all different. Therefore

$$p_{j_1} \cdots p_{j_{f+1}} - p_{g_1} \cdots p_{g_{f+1}} = \left[c \sum_1^{s} p_i\right]_{f+1} + d$$

where c and d, as sums of elements

$$c(i_1, \dots, i_f; i_{f+1}, i_{f+2})$$

and

$$d(i_1, \ldots, i_f; i_{f+1}, i_{f+2}),$$

respectively, satisfy conditions 1 and 2. Since

$$p_1 p_2 \ldots p_{f+1} = \binom{s}{f+1}^{-1} \Big(\sum_{1 \leq i_1 < \ldots < i_{f+1} \leq s} p\, i_1 p\, i_2 \ldots p\, i_{f+1} +$$

$$+ \sum_{1 \leq i_1 < \ldots < i_{f+1} \leq s} (p_1 p_2 \ldots p_{f+1} - p\, i_1 \ldots p\, i_{f+1}) \Big)$$

and

$$\sum p_{i_1} \ldots p_{i_{f+1}} = \Big[c_f \sum_1^s p_i \Big]_{f+1}$$

where

$$c_f = \frac{1}{f+1} \sum_{1 \leq i_1 < \ldots < i_f \leq s} p_{i_1} \ldots p_{i_f}$$

satisfies condition 1, we have proved that

$$D_{2h} = [D_{2(h-1)}\, r_1]_h + V^{(h)}.$$

Since the quadratic form Q_{2h} defined on D_{2h} is non degenerate and the subspaces $[D_{2(h-1)}\, r_1]_h$ and $V^{(h)}$ are orthogonal to each other, we get

$$D_{2h} = [D_{2(h-1)}\, r_1]_h \oplus V^{(h)}$$

and the restrictions of Q_h to $V^{(h)}$ and $[D_{2(h-1)}\, r]_h$ are non-degenerate.

LEMMA 13.—$V^{(h)}$ is irreducible under the group of linear transformations defined by $(P\,S\,(f),\, h)$ or by $(P\,U^{\pm}\,(f),\, h)$.

PROOF.—Since $[D_{2(h-1)}\, r_1]_h$ is invariant under the group of orthogonal transformations defined by $(P\,S\,(f),\, h)$, its orthogonal complement $V^{(h)}$ is also invariant under this group.

If $V^{(h)}$ contains a subspace F irreducible under the orthogonal subgroup defined by $(P\,U^{\pm}\,(f),\, h)$, its orthogonal complement with respecto to $V^{(h)}$, $F^{\perp}$, must be invariant under this subgroup. Hence $F \cap F^{\perp} = 0$ or $F \cap F^{\perp} = F$. If $F \neq V^{(h)}$, by lemma 7, F does not contain any element with a non-zero projection on an index subspace whose index system contains $2h$ different indices. Then all such index subspaces belong to $F^{\perp}$ (see [1], Ch. III, lemma 3) and $F^{\perp} = V^{(h)}$. But, since the restriction of Q_{2h} to $V^{(h)}$ is non-degenerate $F = F^{\perp\perp} = 0$.

THEOREM 1.—The linear groups defined by $(P\,S\,(f),\, h)$, and $(P\,S\,(f),\, h+k)$ induce in $V^{(h)}$ and $[V^{(h)}\, r_1^k]_{h+k}$, respectively, equivalent representations of $P\,S\,(f)$ for

$$h = 1, 2, \ldots, \left[\frac{n}{2}\right]$$

and $2h + k \leqslant n$. If

$$2h + k > n, \quad [V^{(h)}\, r_1^k]_{h+k} = 0.$$

PROOF.—Let $\bar{U}_h$ and $\bar{U}_{h+k}$ be ther linear transformations of D_{2h} and $D_{2(h+k)}$ defined by the element $\bar{U} \in P\,S\,(f)$, i. e., U_h and U_{h+k} are the linear transformations induced in D_{2h} and $D_{2(h+k)}$ by the automorphism σ of $D\,(f)$ associated to $\bar{U}$. If $q_1, q_2, \ldots, q_l$ is a a basis of $V^{(h)}$ and

$$q_j \bar{U}_h = q_j^{\sigma} = \sum_i a_{ji}\, q_i,$$

$$\left([q_j\, r_1^k]_{h+k}\right) \bar{U}_{h+k} = [q_j\, r_1^k]^{\sigma}_{h+k} = [(q_j\, r_1^k)^{\sigma}]_{h+k} = [q_j^{\sigma}\, r_1^k]_{h+k} =$$

$$= \Big[\sum_i a_{ji}\, q_i\, r_1^k\Big]_{h+k} = \sum a_{ji} [q_i\, r_1^k]_{h+k}.$$

Since $V^{(h)}$ is irreducible $[V^{(h)} r_1^k]_{h+k}$ is either o or an isomorphic image of $V^{(h)}$. For the subspace E of $V^{(h)}$ such that

$$\left[E\, r_1^k\right]_{h+k} = \left[E\, r_k\right]_{h+k} = 0$$

must be invariant under the linear group defined by (PS (f, h); but

$$\left[V^{(h)} r_1^k\right]_{h+k} \begin{cases} \neq 0 & \text{if } 2h+k \leq n \\ = 0 & \text{if } 2h+k < n \end{cases}$$

because $V^{(h)}$ contains a non zero element b of the subspace of indices $1, 2, \ldots, 2h$ and

$$\left[b\, r_1^k\right]_{h+k} = \sum_{i_1 < \ldots < i_k} b\, p_{i_1} p_{i_2} \ldots p_{i_k} \neq 0\,,$$

if $2h + k \leqslant n$, where the sum extends over all the combinations of order k of the indices $2h+1, 2h+2, \ldots, n$; whereas, if

$$k > n - 2h\,, \quad \left[b\, r_1^k\right]_{h+k} = \left[b\Big(\sum_{2h+1}^{n} p_j\Big)^k\right]_{h+k} = 0\,.$$

THEOREM 2.—Under the group of linear transformations defined by (PS (f), m) or (P $U^{\pm}$ (f), m), D_{2m} decomposes in a direct sum of irreducible subspaces in the following way,

1) if

$$m \leq \left[\frac{n}{2}\right], \quad D_{2m} = \left[K\, r_1^m\right]_m \oplus \left[V^{(1)} r_1^{m-1}\right]_m \oplus \cdots \oplus \left[V^{(m-1)} r_1\right]_m \oplus V^{(m)};$$

2) if

$$m > \left[\frac{n}{2}\right], \quad D_{2m} = \left[K\, r_1^m\right]_m \oplus \left[V^{(1)} r_1^{m-1}\right]_m \oplus \cdots \oplus \left[V^{(n-m)} r^{2m-n}\right]_m.$$

The space $V^{(h)}$ has dimension $\binom{n}{h}^2 - \binom{n}{h-1}^2$.

PROOF.—To prove 1) we use induction on $m \leqslant \left[\frac{n}{2}\right]$.

If $m = 1$, by lemma 12, $D_2 = K\, r_1 \oplus V^{(1)}$ and

$$\dim V^{(1)} = \dim D_2 - 1 = \binom{n}{1}^2 - 1$$

(cf. [1], Ch. II, th. 1). Moreover, by lemma 13, $K\, r_1$ and $V^{(1)}$ are irreducible.

$$\text{If} \quad D_{2(m-1)} = \left[K\, r_1^{m-1}\right]_{m-1} \oplus \cdots \oplus V^{(m-1)},$$

and

$$\dim V^{(m-1)} = \binom{n}{m-1}^2 - \binom{n}{m-2}^2,$$

then

$$D_{2m} = \left[D_{2(m-1)} r_1\right]_m \oplus V^{(m)} = \Big\{\left[\left[K\, r_1^{m-1}\right]_{m-1} r_1\right]_m + \cdots + \left[\left[r_1^{m-i-1} V^{(i)}\right]_{m-1} r_1\right]_m + \cdots + \left[V^{(m-1)} r_1\right]_m\Big\} \oplus V^{(m)}$$

and it is readily seen that

$$\left[V^{(i)} r_1^{m-i}\right]_m = \left[\left[V^{(i)} r_1^{m-i-1}\right]_{m-1} r_1\right]_m.$$

Hence

$$D_{2m} = \Big\{\left[K\, r_1^m\right]_m + \left[V^{(1)} r_1^{m-1}\right]_m + \cdots + \left[V^{(m-i)} r_1^i\right]_m + \cdots + \left[V^{(m-1)} r_1\right]_m\Big\} \oplus V^{(m)},$$

and it is readily seen that

$$\left[V^{(i)} r_1^{m-i}\right]_m = \left[\left[V^{(i)} r_1^{m-i-1}\right]_{m-1} r_1\right]_m.$$

Hence

$$D_{2m} = \left\{ [K r_1^m]_m + [V^{(1)} r_1^{m-1}]_m + \dots + [V^{(m-i)} r_1^i]_m + \dots \right.$$
$$\left. \dots + [V^{(m-1)} r_1]_m \right\} \oplus V^{(m)}.$$

If

$$[K r_1^m]_m + \dots + [V^{(m-1)} r_1]_m$$

is not a direct sum, there exist elements $a_i \in [V^{(i)} r_1^{m-i}]_m$ such that $A = \sum_1^{m-1} a_i = 0$ with some $a_i \neq 0$. Let k be the greatest index such that $a_k \neq 0$. Since $[V^{(k)} r_1^{m-k}]_m \neq 0$ is irreducible any element of this subspace can be written as linear combination of a_k and a finite number of its images under the group of linear transformations defined by $(P\,U^{\pm}(f), m)$. Let

$$b = [c\, r_1^{m-k}]_m \in [V^{(k)} r_1^{m-k}]_m,$$

where c is an element of the subspace of indices $1, 2, \dots, 2k$, then b is a sum of elements belonging to index subspaces whose index systems contain the subindices $1, 2, \dots, 2k$ only once and $m-k$ indices which appear twice. On the other hand

$$b = \sum_j \alpha_j a_k^{\sigma_j}$$

where the σ_j's are automorphisms of $D(f)$ associated to elements of $P\,U^{\pm}(f)$. Then

$$\sum_j \alpha_j A^{\sigma_j} = \sum_{i=1}^{k} \sum_j \alpha_j a_i^{\sigma_j} = \sum_{i=1}^{k-1} \left(\sum \alpha_j a_i^{\sigma_j} \right) + b = 0.$$

Hence

$$b = -\sum_{i=1}^{k-1} \left(\sum_j \alpha_j a_i^{\sigma_j} \right),$$

which is impossible, fo all the elements

$$a_i^{\sigma_j} \in [V^{(i)} r_1^{m-i}]_m, \quad i \leq k-1;$$

have zero projection on any subspace containing $2k$ indices which appear only once. Therefore

$$D_{2m} = [K r_1^m]_m \oplus [V^{(1)} r_1^{m-1}]_m \oplus \dots \oplus [V^{(m-1)} r_1]_m \oplus V^{(m)}$$

and

$$\dim V^{(m)} = \dim D_{2m} - \dim D_{2(m-1)} = \binom{n}{m}^2 - \binom{n}{m-1}^2.$$

2) If

$$m > \left[\frac{n}{2}\right], \quad \text{then} \quad n - m \leq \left[\frac{n}{2}\right].$$

By theorem 1 we know that the subspaces $[V^{(s)} r_1^{m-s}]_m$ are irreducible and they are different from zero if $2s + m - s = m + s \leqslant n$, i. e., $s \leqslant n - m$. Moreover

$$[K r_1^m]_m + [V^{(1)} r_1^{m-1}]_m + \dots + [V^{(n-m)} r_1^{2m-n}]_m \subset D_{2m}$$

and by the argument used in 1) it can be proved that this sum is direct. Hence

$$\dim \left\{ [K r_1^m]_m \oplus \dots \oplus [V^{(n-m)} r_1^{2m-n}]_m \right\} = \binom{n}{n-m}^2 = \binom{n}{m} = \dim D_{2m}$$

and 2) is proved.

LEMMA 15.—Let σ be the automorphism of $C^+(Q)$ associated to a unitarian similitude U of f. Then the restriction of σ to $D(f)$ is the identity if and only if U is a homotecy of f.

PROOF.—By [1], Ch. II, theorem 3 we know that σ is an inner automorphism of $C^+(Q)$ defined by an element $c_u \in D(f)$. **If the** restriction of τ to $D(f)$ is the identity, c_U must be in the center of $D(f)$; therefore c_U commutes with all the elements c_T which define the inner automorphism associated to the unitarian similitude T. But

$$c_{UT} = c_U c_T = c_T c_U = c_{TU}$$

implies that for any unitarian similitude T, $UT = TU\alpha_L$ and this happens if and only if U is a homotecy; in other words, the coset $\bar{U}$ of U in $PS(f)$ must be in the center of this group which consists of the identity coset, i. e., the set of homotecies of f.

THEOREM 3 —For

$$h = 1, 2, \ldots, \left[\frac{n}{2}\right],$$

the restriction of the group of linear transformations defined by $(PS(f), h)$ to $V^{(h)}$ gives a faithful representation of $PS(f)$.

PROOF.—We start considering $h=1$. Then, $D_2 = K r_1 \oplus V^{(1)}$. Since r_1 is left invariant by the inner automorphisms of $D(f)$, if the automorphism associated to an element $\bar{U} \in PS(f)$ leaves invariant the elements of $V^{(1)}$, σ leaves D_2 invariant elementwise. On the other hand we know (cf. [1], Ch. II, th. 2) that the elements of D_2 generate $D(f)$, therefore σ must be the identical automorphism of $D(f)$. Hence, by the preceding lemma, $\bar{U}$ is the identity of $PS(f)$.

To complete the proof of the theorem we show now that if $V^{(h)}$ is elementwise invariant under σ, $V^{(1)}$ must be also elementwise invariant under this automorphism.

By definition $V^{(h)}$ contains all the elements of the form (4). Let c_1 be the element obtained from (4) taking

$$z_{j_{2i-1} j_{2i}} = p_{j_{2i-1}} - p_{j_{2i}}$$

for every i and let $c_2\,(c_3)$ be the element of the form (4) with

$$z_{j_{2i-1} j_{2i}} = p_{j_{2i-1}} - p_{j_{2i}}$$

for every i and let $c_2\,(c_3)$ be the element of the form (4) with

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for $i < h$ and

$$z_{j_{2h-1} j_{2h}} = u_{j_{2h-1} j_{2h}} \quad (z_{j_{2h-1} j_{2h}} = v_{j_{2h-1} j_{2h}}).$$

Then, since σ is a homogeneous automorphism,

$$(c_1 c_2)^\sigma = c_1^\sigma c_2^\sigma = c_1 c_2 \quad ((c_1 c_3)^\sigma = c_1 c_3)$$

implies that

$$[c_1 c_2]_1^\sigma = [c_1 c_2]_1 \quad ([c_1 c_3]_1^\sigma = [c_1 c_3]_1).$$

But

$$[c_1 c_2]_1 = \Big[(p_{j_{2h-1}} - p_{j_{2h}})\, u_{j_{2h-1} j_{2h}} \prod_{i<h} (p_{j_{2i-1}} - p_{2i})^2\Big]_1 =$$

$$= (2\rho)^h v_{j_{2h-1} j_{2h}} \quad \Big([c_1 c_3]_1 = 2\,(2\rho)^{h-1} u_{j_{2h-1} j_{2h}}\Big).$$

Therefore, for any pair

$$i, j, \quad i \neq j, \quad u_{ij}^\sigma = u_{ij}, \quad v_{ij}^\sigma = v_{ij};$$

and consequently

$$[u_{ij} v_{ij}]_1^\sigma = [u_{ij} v_{ij}]_1,$$

i. e.,

$$\big(-2\,Q(x_i)\,Q(x_j)\,(p_i - p_j)\big)^\sigma = -2\,Q(x_i)\,Q(x_j)\,(p_i - p_j).$$

Hence, if $V^{(h)}$ is elementwise invariant under σ, so it is $V^{(1)}$.

We have proved then that the group $PS(f)$ have faithful irreducible representations on the spaces $V^{(h)}$ of dimension

$$\binom{n}{h}^2 - \binom{n}{h-1}^2,$$

or

$$h = 1, 2, \ldots, \left[\frac{n}{2}\right],$$

and that these representations are still irreducible if we restrict them to P U (f), the projective unitary group, or to $PU^{\pm}(f)$. The linear transformations given by these representations are orthogonal transformations with respect to the non-degenerate quadratic form obtained by restriction of Q_{2h} to $V^{(h)}$.

Any subspace of D (f) irreducible under the automorphisms associated to P S (f) or $PU^{\pm}(f)$ is isomorphic to one of the $V^{(h)}$

$$h = 0, 1, \ldots, \left[\frac{n}{2}\right].$$

REFERENCES

WONENBURGER, M. J.: *The spin representation of the unitary group.* «Rev Mat. His. Am.», 1960.

CRONICA

I REUNION ANUAL DE MATEMATICOS ESPAÑOLES

Como se anunció en el número anterior, durante los días **3** al **6** de octubre se ha celebrado la 1.ª Reunión de Matemáticos Españoles. La Reunión se celebró en el Consejo Superior de Investigaciones Científicas, distribuyéndose las secciones de trabajo en los locales del Instituto «Jorge Juan» y del Instituto de «Investigaciones Estadísticas».

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