# STUDY OF A SEMI-INVOLUTIVE SIMILITUDE

por

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RESUMEN

Dado un espacio vectorial M sobre el cuerpo conmutativo K de característica distinta de 2, consideramos una transformación lineal T de M, cuyo cuadrado es igual a la multiplicación escalar por $\mu \in K$, y suponemos que $\mu$ no es un cuadrado en K. Hallamos una imagen isomorfa del subgrupo del grupo de automorfismos semilineales de M formado por los elementos que conmutan proyectivamente con T y calculamos qué valores puede tomar el determinante de las transformaciones lineales que conmutan o anticonmutan con T. A continuación suponemos que T es una semejanza de razón $\mu$ respecto a una forma cuadrática Q sobre M y determinamos un grupo H isomorfo al subgrupo de semisemejanzas de Q que conmutan proyectivamente con T. Estudiamos después los elementos de H que corresponden a las semejanzas y semejanzas directas. Usando estos resultados demostramos que el centralizador de la clase adjunta de T en el grupo proyectivo de semejanzas y el grupo proyectivo de semejanzas directas posee propiedades distintas que el centralizador en cada uno de estos grupos de la clase adjunta de una involución ortogonal $(2, n-2)$. De esto se deduce que un automorfismo del grupo proyectivo de semejanzas o del grupo proyectivo de semejanzas directas no puede cambiar la clase lateral de una involución ortogonal del tipo $(2. n-2)$ en la clase lateral de T. Cuando se trata del grupo proyectivo de semejanzas directas, este resultado sólo tiene aplicación si $(M : K) = 4m$, pues, en otro caso, T no es una semejanza directa.

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Let M be a left vector space over a field K of characteristic different from 2, Q a non-degenerate quadratic form on M and $(x, y)$ the bilinear form associated to Q.

If T is a similitude of Q of ratio $\mu$ such that $T^2 = \mu$, we want to study the centralizer and projective centralizer of T in the group of proper similitudes of Q, $S^+(Q)$. It will be supposed that $\mu$ is not a square in K since for $\mu = \alpha^2$, $\alpha \in K$, the transformations such as T are well-known.

We will start considering T as a linear transformation of M and finding its centralizer and projective centralizer in the group of semilinear isomorphisms of M, $\Gamma L(M)$. Then it will be supposed that T is a similitude with respect to the quadratic form Q and we will study its centralizers and projective centralizers in the group of semi-similitudes of Q, $\Gamma S(Q)$, the group of similitudes $S(Q)$ and the group of proper similitudes $S^+(Q)$. Finally, in § 3, it will be proved that an automorphism of the projective group of proper similitudes $PS^+(Q)$ can not take the coset of T into the coset of a $(2, n-2)$ orthogonal involution. This result will be used in another paper where the automorphisms of $PS^+(Q)$ will be studied.

We could derive the results of §§ 1 and 2 that we need in § 3 from chapter I, §§ 3, 4, 13 of [1] and chap. I, proposition 4 of [2], but the method followed here is self-contained and easier.

## § 1

The semilinear transformations of M will be written to the right and when the multiplication by an element $\alpha \in K$ is considered as a linear transformation it will be written as $\alpha_L$.

The projective centralizer of a semilinear transformation U in a group G of semilinear transformations of M is denoted by

$$\overline{C}_0(U) = \{S \mid SU = US\alpha_L,\ S \in G\}$$

and the centralizer of U in this group by

$$C_0(U) = \{S \mid SU = US,\ S \in G\}.$$

Let T be a linear transformation of the vector space M such that $T^2 = \mu_L$ and $\mu$ is not a square in K. Then, let $F = K(\theta)$

be the extension of K obtained adjoining the element $\theta$ root of the equation $X^2 - \mu = 0$. The involutive automorphism of F leaving K elementwise invariant and taking $\theta$ into $-\theta$ will be denoted by J.

We make the extension of M, $M_F \approx F \otimes_K M$; therefore $M_F$ is a vector space over F and $M \subset M_F$. If $x_1, x_2, \ldots, x_n$ is a basis of M over K, it is also a basis of $M_F$ over F. Let $T_F$ be the extension of T to a linear transformation of $M_F$. The transformation $T_F^* = T_F \theta_L^{-1}$ verifies $T_F^{*2} = 1_L$; therefore with respect to $T_F^*$, $M_F = M_F^+ \oplus M_F^-$.

LEMMA 1.—Any semilinear transformation $(U, \sigma)$ of automorphism $\sigma$, $U \in \overline{C}_{\Gamma L(M)}(T)$, can be extended in one way and only one to a semilinear transformation of $M_F$ which commutes with $T_F^*$ and induces, therefore, a semilinear transformation of $M_F^+$ in itself.

Proof. Since

$$U \in \overline{C}_{\Gamma L(M)}(T),\ TU = UT\delta_L \quad \text{and} \quad T^2 U = U T^2 \delta_L^2 .$$

But $T^2 = \mu_L$ implies

$$\mu_L U = U \mu_L^\sigma = U \mu_L \delta_L^2, \quad \mu^\sigma = \mu \delta^2.$$

This shows that the automorphism $\sigma$ of K can be extended to an automorphism of F, and this can be done in two different ways,

$$\sigma_F : (\alpha + \beta\theta)^{\sigma_F} = \alpha^\sigma + \delta \beta^\sigma \theta \quad \text{and} \quad \sigma'_F : (\alpha + \beta\theta)^{\sigma'_F} = \alpha^\sigma - \delta \beta^\sigma \theta .$$

To each one of this automorphisms corresponds an extension of $(U, \sigma)$ to a semilinear transformation of M, namely,

$$(U_F, \sigma_F) : (\Sigma (\alpha_i + \beta_i \theta) x_i) U_F = \Sigma \left(\alpha_i^\sigma + \delta \beta_i^\sigma \theta\right)(x_i U),\ \alpha_i, \beta_i \in K,$$

and

$$(U'_F, \sigma'_F) : (\Sigma (\alpha_i + \beta_i \theta) x_i) U'_F = \Sigma \left(\alpha_i^\sigma - \delta \beta_i^\sigma \theta\right)(x_i U).$$



Since

$$TU = UT\delta_L,\quad T_F U_F = U_F T_F \delta_L .$$

and

$$T_F U'_F = U'_F T_F \delta_L .$$

Therefore

$$T_F^* U_F = T_F \theta_L^{-1} U_F = U_F T_F \delta_L \theta_L^{-\sigma_F} = U_F T_F^*$$

and

$$T_F^* U'_F = - U'_F T_F^* .$$

Hence, $U_F$ commutes with $T_F^*$ and induces semilinear transformations $U_F^+$ and $U_F^-$ of $M_F^+$ and $M_F^-$, respectively, in themselves.

This proof implies the

COROLLARY.—A linear transformation U of M commuting with T can be extended to a linear transformation $U_F$ of $M_F$ commuting with $T_F^*$ and a linear transformation of M anticommuting with T ($UT = -UT$) can be extended to a semilinear transformation of $M_F$ of automorphism J commuting with $T_F^*$ or to a linear transformation anticommuting with $T_F^*$.

DEFINITION.—$\varphi$ will be the mapping which takes an element $U \in \overline{C}_{\Gamma L(M)}(T)$ into the semilinear transformation $U_F^+$ of $M_F^+$ induced in this space by the extension of U commuting with $T_F^*$.

It is immediate to see that $\varphi$ is a homomorphism of $\overline{C}_{\Gamma L(M)}(T)$ into $\Gamma L(M_F^+)$. Now we want to prove that $\varphi$ is an isomorphism. For this we need to know the expression of the elements of M in the decomposition $M_F = M_F^+ \oplus M_F^-$. In what follows $x = y \oplus z$ will mean that $y \in M_F^+$ and $z \in M_F^-$.

LEMMA 2. — No element of $M_F^+$ or $M_F^-$ different from zero belongs to M.

Proof. Suppose $0 \neq x \in M$ and $x \in M_F^+$ $(M_F^-)$ Then $xT \in M$ and $xT = xT_F = xT_F^* \theta_L = \theta x\ (-\theta x)$, which is a contradiction, for, if $x \in M$, $\pm \theta x \notin M$.

Hence if $x \in M$ and $x = z \oplus \bar{z}$, $z$ and $\bar{z}$ are both different from 0 if $x \neq 0$.

DEFINITION.—If two elements $z \in M_\nu^+$ and $\bar{z} \in M_\nu^-$ are such that $z + \bar{z} \in M$, each one is called the associate of the other one.

LEMMA 3.—Every element of $M_\nu^+$ ($M_\nu^-$) has a unique associate in M

Proof. Let us see that if

$$z = \Sigma (\alpha_i + \beta_i \theta) x_i \in M_\nu^+,\ \alpha_i, \beta_i \in K,\ \bar{z} = \Sigma (\alpha_i - \beta_i \theta) x_i$$

is its associate. Since $z + \bar{z} = \Sigma 2 \alpha_i x_i \in M$ we have to show that $\bar{z} \in M_\nu^-$.

Because

$$z \in M_\nu^+,\ z T_\nu^* = \Sigma \theta^{-1} (\alpha_i + \beta_i \theta) x_i T = z,$$

and comparing this expression of $z$ with the original one we get

$$\Sigma \alpha_i x_i = \Sigma \beta_i x_i T \text{ and } \Sigma \alpha_i x_i T = \Sigma \mu \beta_i x_i .$$

Hence

$$\bar{z} T_\nu^* = \Sigma \theta^{-1} (\alpha_i - \beta_i \theta) x_i T = - \Sigma (\alpha_i - \theta \beta_i) x_i = - \bar{z} \in M_\nu^- .$$

Conversely, if

$$\bar{z} = \Sigma (\alpha_i' + \beta_i' \theta) x_i \in M_\nu^-$$

we could see that $z = \Sigma (\alpha_i' - \beta_i' \theta) x_i$ is its associate.

The associate is unique, for, if $\bar{z}$, $\bar{z}'$ are both associates of $z$ $(z + \bar{z}) - (z + \bar{z}') = \bar{z} - \bar{z}' \in M$ and lemma 1 implies $\bar{z} = \bar{z}'$, and if $z \neq 0$, $\bar{z} \neq 0$.

LEMMA 4.—The mapping A of each element of $M_\nu^+$ into its associate is a semilineal isomorphism of automorphism J of $M_\nu^+$ onto $M_\nu^-$.



Proof. If $\bar{z}_1$, $\bar{z}_2$ are the associates of $z_1, z_2$, respectively, and $\alpha, \beta \in K$

$$(\alpha z_1 + \beta z_2) + (\alpha \bar{z}_1 + \beta \bar{z}_2) = \alpha (z_1 + \bar{z}_1) + \beta (z_2 + \bar{z}_2) \in M.$$

Therefore $\alpha \bar{z}_1 + \beta \bar{z}_2$ is the associate of $\alpha z_1 + \beta z_2$.

On the other hand if $z \oplus \bar{z} \in M$, $(z \oplus \bar{z}) T \in M$ and

$$(z \oplus \bar{z}) T = (z \oplus \bar{z}) T_\nu = \theta z - \theta \bar{z}$$

implies that $-\theta \bar{z}$ is the associate of $\theta z$. Therefore A is a semilinear transformation of automorphism J. Moreover, A has to be $1-1$ and onto since every element of $M_\nu^+$ or $M_\nu^-$ has an associate which is unique. Hence

$$\dim (M_\nu^+ : F) = \dim (M_\nu^- : F) = \tfrac{1}{2} \dim (M : K).$$

LEMMA 5.—Every semilinear transformation $(U, \sigma)$ of $M_\nu$ commuting with $T_\nu^*$ and leaving M invariant is completely determined by its restriction to $M_\nu^+$. Moreover, if $\sigma$ is its automorphism $\sigma J = J \sigma$.

Proof. U leaves invariant the subspaces $M_\nu^+$ and $M_\nu^-$, besides if $x = z \oplus \bar{z} \in M$, $x U = z U \oplus \bar{z} U \in M$. Therefore $\bar{z} U = z U A = \bar{z} A^{-1} U A$. Hence if $U^+$ is the restriction of U to $M_\nu^+$ and $y = y^+ \oplus y^-$, $y U = y^+ U^+ \oplus y^- A^{-1} U^+ A$.

Since U is a semilinear transformation of automorphism $\sigma$, the automorphism of $A^{-1} U^+ A$, $J \sigma J = \sigma$ and therefore $J \sigma = \sigma J$.

THEOREM 1.—Let T be a $n \times n$ matrix with entries in $K$, such that $T^2 = \mu I_n$ and $\mu$ is not a square in K. Then the determinant of a matrix commuting (anticommuting) with T is of the form $\alpha^2 - \beta^2 \mu$ $((-1)^{n/2} (\alpha^2 - \beta^2 \mu))$.

Proof. We consider T as a linear transformation of the vector space M over K, $\dim (M : K) = n = 2m$. The determinant of the matrix of the transformation T is equal to the determinant of its extension to a linear transformation of $M_\nu$.

If U is a linear transformation of M commuting (anticommuting) with T, its extension to a linear transformation of $M_\nu$ commutes (anticommutes) with $T_\nu^*$. If we take a basis $z_1, z_2, \ldots, z_m$ for $M_\nu^+$, their associates form a basis of $M_\nu^-$. The matrix of

$U_F$ ($U'_F$) with respect to the basis $z_1, \dots, z_m, \bar{z}_1, \dots, \bar{z}_m$ of $M_F$, expressed in $m \times m$ blocks, has the form

$$C = \begin{pmatrix} B & O \\ O & B' \end{pmatrix} \left( D = \begin{pmatrix} O & B \\ B' & O \end{pmatrix} \right), \text{ where } B = (b_{ij}),\ B' = (b'_{ij}),\ b_{ij} \in F.$$

Then

$$\det. C = \det. B \det. B' = (\alpha + \beta\theta)(\alpha - \beta\theta) = \alpha^2 - \beta^2 \mu$$

$$(\det. D = (-1)^m \det. B \det. B' = (-1)^m (\alpha^2 - \beta^2 \mu))$$

THEOREM 2.— $\varphi$ is an isomorphism of $\overline{C}_{\Gamma L(M)}(T)$ onto the group of semilinear transformations of $M_F^+$ whose automorphism commutes with J.

Proof. Suppose that $U \in \overline{C}_{\Gamma L(M)}(T)$ is taken by $\varphi$ into the identity transformation of $M_F^+$. This means that the restriction $U_F^+$ to $M_F^+$ of the extension $U_F$ of U to $M_F$ commuting with $T_F^*$ is the identity. Since $U_F$ leaves M invariant,

$$y U_F = (y^+ \oplus y^-) U_F = y^+ U_F^+ \oplus y^- A^{-1} U_F^+ A = y^+ \oplus y^- = y.$$

Therefore $U_F = 1_L$ and $U = 1_L$, which proves that $\varphi$ is an isomorphism.

Now let us study the image of $\overline{C}_{\Gamma L(M)}(T)$ under the isomorphism $\varphi$. If $(L^+, \sigma)$ is a semilinear transformation which belongs to this image, $L^+$ is the restriction to $M_F^+$ of a semilinear transformation $(L, \sigma)$ of $M_F$ leaving M invariant and commuting with $T_F^*$. We have seen in lemma 5 that this is possible if and only if $\sigma J = J \sigma$. On the other hand since $(L, \sigma)$ commutes with $T_F^*$ and $\sigma$ commutes with J, L commutes projectively with $T_F = T_F^* \theta_L$. Hence the restriction $L_1$ of L to M belongs to $\overline{C}_{\Gamma L(M)}(T)$ and $(L^+, \sigma)$ is the image of $L_1$ under $\varphi$.

## § 2

If Q is a non degenerate quadratic form on the vector space M over the field K, the semisimilitudes $(S, \sigma)$ of ratio $\rho$ of Q



are the semilinear transformations of M of automorphism $\sigma$ satisfying

$$Q(xS) = \rho\, Q(x)^\sigma \qquad \text{for every} \qquad x \in M.$$

When $\sigma$ is the identity we say that S is a similitude. The orthogonal transformations are the similitudes of ratio 1. If

$$\dim (M : K) = n = 2m,$$

it is said that S is a proper similitude of ratio $\rho$ if the determinant of the matrix of the trasformaion S is equal to $\rho^m$.

Let us suppose that the linear transformation T, such that $T^2 = \mu_L$, is a similitude of Q of ratio $\rho$. Let $Q_F$ be the extension of Q to $M_F$ and $(x, y)_F$ its associated bilinear form, which is the extension to $M_F$ of $(x, y)$. That is

$$\left( \sum_i (\alpha_i + \beta_i \theta) x_i, \sum_j (\alpha_j' + \beta_j' \theta) x_j \right)_F =$$
$$= \sum_{ij} (\alpha_i + \beta_i \theta)(\alpha_j' + \beta_j' \theta)(x_i, x_j).$$

Then $Q_F$ is also non degenerate.

LEMMA 6.—The extension of a similitude $(S, \sigma)$ of Q of ratio $\rho$ such that $S \in \overline{C}_{\Gamma L(M)}(T)$ to a semilinear transformation of $M_F$ is a semisimilitude of ratio $\rho$ of $Q_F$.

The proof is straightforward.

In particular $T_F$ is a similitude of ratio $\mu$ of $Q_F$ and $T_F^*$ is an orthogonal transformation. Therefore the subspaces $M_F^+$ and $M_F^-$ are orthogonal to each other with respect to $(x, y)_F$ and the restrictions $Q_F^+$ and $Q_F^-$ of Q to these subspaces are non-degenerate.

LEMMA 7.—If $z_1, z_2 \in M_F^+$ and $\bar{z}_1, \bar{z}_2$ are their associates

$$(z_1, z_2)_F = (\bar{z}_1, \bar{z}_2)_F^J.$$

Proof. Let

$$z_1 = \Sigma (\alpha_i + \beta_i \theta) x_i$$

and

$$z_2 = \Sigma (\alpha_j' + \beta_j' \theta) x_j.$$

Then

$$(z_1, z_2)_F = \sum_{i,j} (\alpha_i + \beta_i \theta)(\alpha_j' + \beta_j' \theta)(x_i, x_j)$$

and

$$(\bar{z}_1, \bar{z}_2)_F = \sum_{i,j} (\alpha_i - \beta_i \theta)(\alpha_j' - \beta_j' \theta)(x_i, x_j) =$$

$$= \Big( \sum_{i,j} (\alpha_i + \beta_i \theta)(\alpha_j' + \beta_j' \theta)(x_i, x_j) \Big)^J = (z_1, z_2)_F^J .$$

LEMMA 8.—Let $(U^+, \tau)$ be a semimilitude of ratio $r$ of $Q_F^+$ and $\tau J = J \tau$. Then $U^+$ can be extended to a semisimilitude of $Q_F$ leaving M invariant if and only if $r \in K$.

Proof. By lemma 5 there exists only one extension of $U^+$ to a semilinear transformation of $M_F$ leaving M and $M_F^+$ invariant. It has to be checked that this extension is a semisimilitude. We have

$$(y\,U, z\,U)_F = (y^+ U^+ \oplus y^- A^{-1} U^+ A, z^+ U^+ \oplus z^- A^{-1} U^+ A)_F =$$

$$= (y^+ U^+, z^+ U^+)_F + (y^- A^{-1} U^+ A, z^- A^{-1} U^+ A)_F =$$

$$= r\,(y^+, z^+)_F^\tau + (y^- A^{-1} U^+, z^- A^{-1} U^+)_F^J =$$

$$= r\,(y^+, z^+)_F^\tau + \big(r\,(y^- A^{-1}, z^- A^{-1})_F^\tau\big)^J = r\,(y^+, z^+)_F^\tau + r^J (y^-, z^-)_F^\tau .$$

This is equal to $r\,(y, z)_F^\tau$ if and only if $r = r^J$. Hence $r \in K$.

THEOREM 3.—The restriction of $\varphi$ to the group $\overline{C}_{\Gamma S(Q)}(T)$, proyective centralizer of T in the group of semisimilitudes of Q, is an isomorphism of this group onto the subgroup $R(Q_F^+, K)$ of semisimilitudes of $Q_F^+$ of ratio $\rho \in K$ whose automorphism $\tau$ commutes with J.

Proof. By lemma 6 we know that the extension to $M_F$ of a semisimilitude $S \in \overline{C}_{\Gamma S(Q)}(T)$ is a semisimilitude of $Q_F$ of the same ratio $\rho \in K$ and there is only one extension $S_F$ which commutes with $T_F^*$. Its restriction $S_F^+$ to $M_F^+$ is a semisimilitude of $Q_F^+$ of the same ratio. Therefore $\varphi$ is an isomorphism of $\overline{C}_{\Gamma S(Q)}(T)$ into the subgroup $R(Q_F^+, K)$. Now lemma 8 shows that this is an isomorphism onto.



COROLLARY 1. — $\varphi$ induces an isomorphism of the group $\overline{C}_{S(Q)}(T)$, projective centralizer of T in the group of similitudes of Q, onto the subgroup of semisimilitudes of $Q_F^+$ of ratio $\rho \in K$ whose automorphism is the identity or J.

Proof. The image of an element of $\overline{C}_{S(Q)}(T)$ is a semisimilitude of $Q_F^+$ of the same ratio whose automorphism $\tau$ is an extension to F of the automorphism identity of K. Therefore $\tau$ is the identity or J.

On the other hand any semisimilitude $S_F^+$ of $Q_F^+$ of ratio $\rho \in K$ whose automorphism is the identity or J can be extended to a semisimilitude of $M_F$ with the same automorphism and leaving M invariant. Hence this extension induces on M a similitude whose image by $\varphi$ is $S_F^+$.

COROLLARY 2. — $\varphi$ induces an isomorphism of the group $C_{O^+(Q)}(T)$, centralizer of T in the group of rotations of Q, onto the orthogonal group of $Q_F^+$, $O(Q_F^+)$.

Proof. If $U \in C_{O^+(Q)}(T)$, its extension $U_F$ to $M_F$ commuting with $T_F^*$ is an orthogonal transformation by the corollary of lemma 1.

Now let $U_F^+$ be any orthogonal transformation of $Q_F^+$ and define $U_F$ by $y\,U_F = y^+ U_F^+ \oplus y^- A^{-1} U_F^+ A$. Let $z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m$ be a basis of $M_F$ formed by the elements of a basis of $M_F^+$ and their associates. The matrix C of $U_F$ with respect to this basis, expressed in $m \times m$ blocks, has the form

$$C = \begin{pmatrix} B & O \\ O & B^J \end{pmatrix} \text{ where } B = (b_{ij}),\ B^J = (b_{ij}^J) .$$

Since B is the matrix of an orthogonal transformation, its determinant is $\pm 1$. In either case

$$\det. C = \det. B \det. B^J = \det. B\,(\det. B)^J = 1,$$

which shows that $U_F$ is a rotation. Hence its restriction to M is a rotation of Q mapped by $\varphi$ into $U_F^+$.

## § 3

We will denote by $\overline{S}$ the coset of a similitude S of Q in the projective group of similitudes $PS(Q) \approx S(Q)/K^*$ or the projective group of proper similitudes $PS^+(Q) \approx S^+(Q)/K^*$, where $K^*$ is the multiplicative group of non-zero elements of K. If Z is any group, $Z'$ will be its commutator group. The similitude T will be as before.

THEOREM 4.—If $\dim (M : K) \geqslant 6$, the center of the commutator group of $C_{PS^+(Q)}(\overline{T})$ (or $C_{PS(Q)}(\overline{T})$) is the coset of the identity.

Proof.

$$C_{PS^+(Q)}(\overline{T}) \approx \overline{C}_{S^+(Q)}(T)/K^*.$$

By corollary 1 of theorem 3 we know that $\varphi$ is an isomorphism of $\overline{C}_{S^+(Q)}(T)$ onto a subgroup G of $R(Q_F^+, K)$, and by corollary 2 we know that $G \supset O(Q_F^+)$.

The commutator group

$$(\overline{C}_{S^+(Q)}(T)/K^*)' \approx (\overline{C}_{S^+(Q)}(T))'/K^* \cap (\overline{C}_{S^+(Q)}(T))'.$$

This last group is mapped by $\varphi$ onto $G'/K^* \cap G'$ which contains to $\Omega(Q_F^+)/K^* \cap \Omega(Q_F^+)$, where $\Omega(Q_F^+)$ is the commutator group of $O(Q_F^+)$.

There are no similitudes or semisimilitudes different from the scalar multiplications $(\alpha + \beta\theta)_L$ commuting projectively with the commutator $\Omega(Q_F^+)$ if $\dim (M_F^+ : F) \geqq 3$. Since the commutator $G'$ only contains orthogonal transformations, the elements of $G'$ commuting projectively with $\Omega(Q_F^+)$ are in $K^* \cap G'$. Therefore $K^* \cap G'$ is the center of $G'/K^* \cap G'$.

The same proof goes over if we consider $PS(Q)$ instead of $PS^+(Q)$, substituting $R(Q_F^+, K)$ for G.

LEMMA 9. — If $\dim (M : K) = n > 4$, let U be an $(2, n-2)$ orthogonal involution of Q. Then the center of the commutator group of $C_{PS^+(Q)}(\overline{U})$ (or $C_{PS(Q)}(\overline{U})$) is a subgroup which contains properly the coset of the identity.



Proof. Let $N^+$ and $N^-$ be the plus —and minus— spaces of U. Then $M = N^+ \oplus N^-$, $\dim (N^+ : K) = 2$, $\dim (N^- : K) = n-2$ and any similitude commuting projectively with U commutes with U. Hence

$$C_{PS^+(Q)}(\overline{U}) \approx C_{S^+(Q)}(U)/K^*$$

and the elements of $C_{S^+(Q)}(U)$ induce similitudes in $N^+$ and $N^-$ with respect to the restriction $Q_{N^+}, Q_{N^-}$ of Q to these subspaces. The commutator

$$(C_{PS^+(Q)}(\overline{U}))' \approx (C_{S^+(Q)}(U))'/K^* \cap (C_{S^+(Q)}(U))$$

and $(C_{S^+(Q)}(U))'$ contains $\Omega(Q_{N^+}) \otimes \Omega(Q_{N^-})$ and is contained in $O^+(Q_{N^+}) \otimes O^+(Q_{N^-})$. Therefore the center of $(C_{S^+(Q)}(U))'$ contains elements of the form $S \otimes I_{N^-}$, where $S \in O^+(Q_{N^+})$) and $I_{N^-}$ is the identity element of $O(Q_{N^-})$, because $O^+(Q_{N^+})$ is an abelian group. But, if S is not the identity element of $O(Q_{N^+})$,

$$S \otimes I_{N^-} \notin K^* \cap (C_{T^+(Q)}(U))'.$$

The proof holds if we substitute $PS(Q)$ for $PS^+(Q)$ and $S(Q)$ for $S^+(Q)$.

From theorem 4 and lemma 9 follows.

THEOREM 5.—If $\dim (M : F) \geqq 6$, there are no automorphisms of $PS^+(Q)$ ($PS(Q)$) taking the coset of the similitude T of ratio $\mu$, such that $T^2 = \mu_L$, $\mu$ not a square in K, into the coset of a $(2, n-2)$ orthogonal involution.

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