TWISTED FROBENIUS-SCHUR INDICATORS OF FINITE SYMPLECTIC GROUPS

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1. Introduction

In [8], R. Gow proves the following theorem.

Theorem 1.1. *Let $G = GL(n, \mathbb{F}_q)$, where q is odd. Let G^+ be the split extension of G by the transpose-inverse automorphism. That is,*

$$G^+ = \langle G, \tau \mid \tau^2 = 1, \tau^{-1} g \tau = {}^t g^{-1} \text{ for all } g \in G \rangle.$$

Then all complex irreducible representations of G^+ can be obtained in the field of real numbers.

Denote by $\varepsilon(\pi)$ the classical Frobenius-Schur indicator of the irreducible representation (π, V) with character χ. That is, $\varepsilon(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$, and Frobenius and Schur proved that $\varepsilon(\pi) = 1$ if (π, V) is a real representation, $\varepsilon(\pi) = -1$ if χ is real-valued, but π is not a real representation, and $\varepsilon(\pi) = 0$ otherwise. Then the conclusion of Theorem 1.1 is equivalent to the statement that every irreducible representation π of G^+ satisfies $\varepsilon(\pi) = 1$.

Gow obtained the following intriguing result from Theorem 1.1.

Corollary 1.1. *The sum of the degrees of the irreducible representations of $GL(n, \mathbb{F}_q)$ is equal to the number of symmetric elements in $GL(n, \mathbb{F}_q)$.*

Theorem 1.1 only implies Corollary 1.1 for q odd, but Klyachko [16] obtained Corollary 1.1 for any q by obtaining a *model* for $GL(n, \mathbb{F}_q)$. Different proofs of Klyachko's main theorem in [16, Theorem A] were also obtained by Inglis and Saxl [13] and by Howlett and Zworestine [11]. I.G. Macdonald also proves Corollary 1.1 for any q by directly computing the sum of the degrees of the characters of $GL(n, \mathbb{F}_q)$ using symmetric functions [17, Ch. IV.6, Ex. 5].

Several years after the papers of Gow and Klyachko, Kawanaka and Matsuyama [15] developed the notion of a *twisted* Frobenius-Schur indicator. If (π, V) is a complex irreducible representation of a finite group G, the twisted Frobenius-Schur indicator of π depends on an order two automorphism ι of G, and is denoted $\varepsilon_\iota(\pi)$. Let χ be the character associated to the representation π. Then Kawanaka and Matsuyama define $\varepsilon_\iota(\pi)$ as follows:

$$\varepsilon_\iota(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi(g \, {}^\iota g).$$

If ι is the identity automorphism of G, then $\varepsilon(\pi) = \varepsilon_\iota(\pi)$. One of the main results of [15] is that $\varepsilon_\iota(\pi)$ is a useful indicator for the irreducible representation (π, V), as we now explain. For a chosen basis of V, let $R(g)$ denote the matrix for $\pi(g)$ with respect to this basis. Then the following holds.

$$\varepsilon_\iota(\pi) = \begin{cases} 1 & \text{if a basis for } V \text{ may be chosen such that} \\ & R({}^\iota g) = \overline{R(g)} \text{ for all } g \in G \\ -1 & \text{if } \chi({}^\iota g) = \overline{\chi(g)} \text{ for all } g \in G, \text{ but there does not exist} \\ & \text{a basis for } V \text{ such that } R({}^\iota g) = \overline{R(g)} \text{ for all } g \in G \\ 0 & \text{otherwise.} \end{cases}$$

One of the implications of Theorem 1.1 is that every irreducible π of $GL(n, \mathbb{F}_q)$ satisfies $\varepsilon_\iota(\pi) = 1$, where here ${}^\iota g = {}^t g^{-1}$. By applying the twisted version of the involution formula, as we will see in Proposition 2.1, we are able to obtain Corollary 1.1.

Before the paper of Kawanaka and Matsuyama, Gow also proved [9] the following results about the symplectic group over a finite field.

Theorem 1.2. *Let $G = Sp(2n, \mathbb{F}_q)$ with q odd. Each non-faithful real-valued irreducible character of G is the character of a real representation, whereas each faithful real-valued irreducible character of G has Schur index 2 over the real numbers.*

Corollary 1.2. *When $q \equiv 1(\text{mod } 4)$, the sum of the degrees of the irreducible complex characters of the symplectic group $Sp(2n, \mathbb{F}_q)$ is given by*

$$q^{n(n+1)/2}(q^n + 1) \cdots (q + 1).$$

Gow [9, p. 251] notes that Corollary 1.2 "probably holds when $q \equiv 3(\text{mod } 4)$, but our method of proof does not yield such a result". It is this open case that motivates the work in this paper.

For odd q, since $PSp(2n, \mathbb{F}_q)$ is simple (except for $n = 1$, $q = 3$, which can be handled separately), an irreducible representation π of $Sp(2n, \mathbb{F}_q)$ is faithful if and only if its central character ω_π satisfies $\omega_\pi(-I) = -1$. Therefore, the content of Theorem 1.2 is that any irreducible representation π of $Sp(2n, \mathbb{F}_q)$ whose character is real-valued satisfies $\varepsilon(\pi) = \omega_\pi(-I)$.

Define ι to be the order 2 automorphism for $Sp(2n, \mathbb{F}_q)$ which conjugates elements by a certain skew-symplectic element:

$$ {}^\iota g = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix} g \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}. \tag{1}$$

When $q \equiv 1(\text{mod } 4)$, the automorphism ι defined above is inner by an element whose square is $-I$, and from this fact it will follow from Lemma 2.1 that $\varepsilon_\iota(\pi) = \omega_\pi(-I)\varepsilon(\pi)$ for any irreducible π of G. For $q \equiv 1(\text{mod } 4)$, every irreducible character of $Sp(2n, \mathbb{F}_q)$ is real-valued, and so by Theorem

1.2, we have $\varepsilon_\iota(\pi) = 1$ for every irreducible representation π, and Corollary 1.2 follows by applying a counting argument.

When $q \equiv 3(\mathrm{mod}\ 4)$, the automorphism ι is not an inner automorphism, and there are irreducible characters of $Sp(2n, \mathbb{F}_q)$ which are not real-valued. However, the main result of this paper is the following, which covers all odd q, thus generalizing Theorem 1.2 and Corollary 1.2.

Theorem 1.3. *Let q be odd, and ι the automorphism of $Sp(2n, \mathbb{F}_q)$ as in (1). Then every irreducible representation π of $Sp(2n, \mathbb{F}_q)$ satisfies $\varepsilon_\iota(\pi) = 1$. The sum of the degrees of the irreducible complex characters of $Sp(2n, \mathbb{F}_q)$ is given by*

$$q^{n(n+1)/2}(q^n + 1) \cdots (q + 1).$$

The main idea in proving Theorem 1.3 for the case $q \equiv 3(\mathrm{mod}\ 4)$ is to consider the following group, which contains $G = Sp(2n, \mathbb{F}_q)$ as an index 2 subgroup:

$$Sp(2n, \mathbb{F}_q)^{\iota, -I} = \langle G, \tau \mid \tau^2 = -I, \tau^{-1} g \tau = {}^\iota g \text{ for all } g \in G \rangle.$$

We will prove that every irreducible ϕ of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ satisfies $\varepsilon(\phi) = \omega_\phi(-I)$, and this is the main statement needed to obtain Theorem 1.3, as described in Proposition 2.2. The method that is used, like in [8, 9], is an induction argument using the Berman-Witt generalization of the Brauer induction theorem. The bulk of the work for this argument is in the analysis of maximal $\mathbb{R}$-elementary subgroups at 2, in Section 5.

We also obtain results for the group of similitudes $GSp(2n, \mathbb{F}_q)$ for q odd. Let μ be the similitude character, and ι the inner automorphism of $GSp(2n, \mathbb{F}_q)$ which conjugates by the skew-symplectic element as in (1). Define σ to be the order 2 automorphism of $GSp(2n, \mathbb{F}_q)$ which acts as ${}^\sigma g = \mu(g)^{-1} \cdot {}^\iota g$. Then the main result for this group is the following, with the rather surprising result for the sum of the character degrees.

Theorem 1.4. *Let q be odd, and let σ be the automorphism of $GSp(2n, \mathbb{F}_q)$ as defined above. Then every irreducible π of $GSp(2n, \mathbb{F}_q)$ satisfies $\varepsilon_\sigma(\pi) = 1$. The sum of the degrees of the irreducible characters of $GSp(2n, \mathbb{F}_q)$ is equal to the number of symmetric matrices in $GSp(2n, \mathbb{F}_q)$.*

2. Twisted Frobenius-Schur Indicators

If (π, V) is an irreducible complex representation of a finite group G, call it a *real representation* if there exists a basis for V such that for any $g \in G$, all entries of the matrix for $\pi(g)$ with respect to this basis are in the field of real numbers. For an irreducible χ of G, define the *Frobenius-Schur indicator* of χ, written $\varepsilon(\chi)$, to be $\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$. Frobenius and Schur [6] proved the following theorem, which gives a useful meaning to $\varepsilon(\chi)$.

Theorem 2.1. *Let χ be the character of a complex irreducible representation of a finite group G. Then*

$$\varepsilon(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is the character of a real representation} \\ -1 & \text{if } \chi \text{ is real-valued but not the character of} \\ & \text{a real representation} \\ 0 & \text{otherwise.} \end{cases}$$

In 1990, Kawanaka and Matsuyama [15] generalized the notion of Frobenius-Schur indicators to include a twist by an order two automorphism of G, $\iota : G \to G$. For a complex irreducible character χ of G, define

$$\varepsilon_\iota(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g\, {}^\iota g),$$

where we are writing the automorphism ι as acting on the left. Kawanaka and Matsuyama proved the following generalization of Frobenius and Schur's Theorem 2.1. For a complex representation (π, V) of a finite group G, if a basis is chosen for V, let $R(g)$ denote the matrix for $\pi(g)$ with respect to this basis.

Theorem 2.2. *Let χ be the character of a complex irreducible representation (π, V) of the group G, $\iota : G \to G$ an order 2 automorphism of G, and $\varepsilon_\iota(\chi)$ defined above. Then*

$$\varepsilon_\iota(\chi) = \begin{cases} 1 & \text{if a basis for } V \text{ may be chosen such that} \\ & R({}^\iota g) = \overline{R(g)} \text{ for all } g \in G \\ -1 & \text{if } \chi({}^\iota g) = \overline{\chi(g)} \text{ for all } g \in G, \text{ but there does not exist} \\ & \text{a basis for } V \text{ such that } R({}^\iota g) = \overline{R(g)} \text{ for all } g \in G \\ 0 & \text{otherwise.} \end{cases}$$

We call $\varepsilon_\iota(\chi)$ the *twisted Frobenius-Schur indicator* of χ.

We now take a slightly different point of view of the situation. If (π, V) is a complex representation of a finite group G, then the contragredient of π is the representation of G acting on the dual space V^*, written $\hat{\pi}$, with action $\hat{\pi}(g)(l(v)) = l(\pi(g^{-1})v)$, where $v \in V$ and $l \in V^*$. If χ is the character of π, then $\overline{\chi}$ is the character of $\hat{\pi}$. Let (π, V) be a complex irreducible representation of a finite group G such that $\pi \cong \hat{\pi}$, which is equivalent to the character χ of π being real-valued. Then $\pi \cong \hat{\pi}$ if and only if there is a nondegenerate bilinear form $B : V \times V \to \mathbb{C}$ such that

$$B(\pi(g)u, \pi(g)v) = B(u, v) \text{ for all } g \in G,\ u, v \in V.$$

By an application of Schur's Lemma, B is unique up to scalar, which implies

$$B(u, v) = \epsilon(\pi) B(v, u) \tag{2}$$

for a complex number $\epsilon(\pi)$ which evidently satisfies $\epsilon(\pi)^2 = 1$. Define $\epsilon(\pi) = 0$ if $\pi \not\cong \hat{\pi}$. Then $\varepsilon(\pi)$ is exactly the Frobenius-Schur indicator $\varepsilon(\chi)$ in Theorem 2.1. This is proven in a much more general setting by Bump and

Ginzburg [1], but we now only state the generalization that we need, which is the case corresponding to Theorem 2.2 of Kawanaka and Matsuyama.

Let ι be an order 2 automorphism of the finite group G, and let (π, V) be a complex irreducible representation of G such that ${}^{\iota}\pi \cong \hat{\pi}$, where ${}^{\iota}\pi(g) = \pi({}^{\iota}g)$. Note that if χ is the character of π and ${}^{\iota}\chi$ the character of ${}^{\iota}\pi$, then ${}^{\iota}\pi \cong \hat{\pi}$ is equivalent to the statement that ${}^{\iota}\chi = \overline{\chi}$. Similar to the untwisted case above, we have this isomorphism if and only if there is a nondegenerate bilinear form B_ι, unique up to scalar by Schur's Lemma, satisfying

$$B_\iota(\pi(g)u, {}^{\iota}\pi(g)v) = B_\iota(u, v) \text{ for all } g \in G,\ u, v \in V.$$

So there is a constant $\varepsilon_\iota(\pi)$ satisfying

$$B_\iota(u, v) = \varepsilon_\iota(\pi) B_\iota(v, u) \text{ and } \varepsilon_\iota(\pi)^2 = 1. \tag{3}$$

Letting $\varepsilon_\iota(\pi) = 0$ when ${}^{\iota}\pi \not\cong \hat{\pi}$, we have that $\varepsilon_\iota(\pi)$ is exactly the twisted Frobenius-Schur indicator $\varepsilon_\iota(\chi)$ of Kawanaka and Matsuyama as in Theorem 2.2. This point of view of twisted Frobenius-Schur indicators is sometimes more convenient in proofs.

The following proposition, which generalizes the Frobenius-Schur involution formula, relates twisted Frobenius-Schur indicators to combinatorial information. This result is implicit in Kawanaka and Matsuyama's work [15], and a proof of a more general statement appears in Bump and Ginzburg's paper [1, Proposition 1, Theorem 2].

Proposition 2.1. *Let G be a finite group and ι an order 2 automorphism of G.*
(i) For any $h \in G$,

$$\sum_{\chi \in \mathrm{Irr}(G)} \varepsilon_\iota(\chi)\chi(h) = |\{g \in G \mid g\, {}^{\iota}g = h\}|.$$

(ii) $\varepsilon_\iota(\chi) = 1$ for every $\chi \in Irr(G)$ if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = |\{g \in G \mid {}^{\iota}g = g^{-1}\}|.$$

If ι is an inner automorphism of G, then we may say precisely how $\varepsilon(\pi)$ and $\varepsilon_\iota(\pi)$ are related. If ι is inner given by ${}^{\iota}g = h^{-1}gh$, then since ι is of order two, we have h^2 is in the center of G.

Lemma 2.1. *Let ι be an inner automorphism of order 2 of G, given by ${}^{\iota}g = h^{-1}gh$, where $h^2 = z$ is in the center of G. Then for any $\chi \in \mathrm{Irr}(G)$ with central character ω_χ,*

$$\varepsilon_\iota(\chi) = \omega_\chi(z)\varepsilon(\chi).$$

Proof. We have

$$\varepsilon_\iota(\chi) = \frac{1}{|G|}\sum_{g \in G} \chi(g\, {}^{\iota}g) = \frac{1}{|G|}\sum_{g \in G} \chi(z^{-1}(gh)^2) = \frac{\overline{\chi(z)}}{\chi(1)}\varepsilon(\chi).$$

Now if χ is real-valued, then $\overline{\frac{\chi(z)}{\chi(1)}} = \frac{\chi(z)}{\chi(1)} = \omega_\chi(z)$. If χ is not real-valued, then $\varepsilon(\chi) = 0$, and so $\varepsilon_\iota(\chi) = 0$ from the above calculation. Thus in either case we have $\varepsilon_\iota(\chi) = \omega_\chi(z)\varepsilon(\chi)$. $\square$

Now we consider extensions of the group G using a given order 2 automorphism ι. For any order 2 element z in the center of G, we define the following group, which contains G as an index 2 subgroup:

$$G^{\iota,z} = \langle G, \tau \mid \tau^2 = z, \tau^{-1} g \tau = {}^\iota g \text{ for all } g \in G \rangle.$$

The idea is to get information about the $\varepsilon_\iota(\chi)$'s for $\chi \in \mathrm{Irr}(G)$ by studying the $\varepsilon(\psi)$'s for $\psi \in \mathrm{Irr}(G^{\iota,z})$. Let us start with an irreducible representation (π, V) of G with character χ, and define π^+ to be the representation π of G induced to $G^{\iota,z}$, and call its character χ^+. There is a nice criterion for χ^+ to be irreducible.

Lemma 2.2. *π^+ is an irreducible representation of $G^{\iota,z}$ if and only if $\pi \not\cong {}^\iota\pi$.*

Proof. By Frobenius reciprocity, $\langle \chi^+, \chi^+ \rangle_{G^{\iota,z}} = \langle \chi, \chi^+|_G \rangle_G$. By direct calculation, we see that $\chi^+(g) = \chi(g) + {}^\iota\chi(g)$ for $g \in G$. So we have

$$\begin{aligned} \langle \chi^+, \chi^+ \rangle_{G^{\iota,z}} &= \langle \chi, \chi^+|_G \rangle_G = \langle \chi, \chi + {}^\iota\chi \rangle_G \\ &= 1 + \langle \chi, {}^\iota\chi \rangle = \begin{cases} 2 & \text{if } \pi \cong {}^\iota\pi \\ 1 & \text{if } \pi \not\cong {}^\iota\pi \end{cases}. \ \square \end{aligned}$$

Lemma 2.3. *For any $\chi \in \mathrm{Irr}(G)$ with central character ω_χ, we have*

$$\frac{1}{|G^{\iota,z}|} \sum_{g \in G^{\iota,z}} \chi^+(g^2) = \varepsilon(\chi) + \omega_\chi(z)\varepsilon_\iota(\chi).$$

Proof. We first split the sum into the two cosets of G in $G^{\iota,z}$, G and $G\tau$. Then note that $(g\tau)^2 = g\tau g\tau = gz\tau^{-1}g\tau = zg\,{}^\iota g$. Finally we apply the fact that $\chi^+(g) = \chi(g) + {}^\iota\chi(g)$ for $g \in G$:

$$\begin{aligned} \frac{1}{|G^{\iota,z}|} \sum_{g \in G^{\iota,z}} \chi^+(g^2) &= \frac{1}{|G^{\iota,z}|} \Big(\sum_{g \in G} \chi^+(g^2) + \sum_{g \in G} \chi^+(zg\,{}^\iota g) \Big) \\ &= \frac{1}{2|G|} \Big(\sum_{g \in G} (\chi(g^2) + \chi({}^\iota g^2)) + \sum_{g \in G} (\chi(zg\,{}^\iota g) + \chi(z\,{}^\iota g g)) \Big). \end{aligned}$$

But now as g runs over all elements of G, so does ${}^\iota g$. So the sum is

$$= \frac{1}{2|G|} \Big(2 \sum_{g \in G} \chi(g^2) + 2 \sum_{g \in G} \chi(zg\,{}^\iota g) \Big)$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g^2) + \frac{\omega_\chi(z)}{|G|} \sum_{g \in G} \chi(g\,{}^\iota g) = \varepsilon(\chi) + \omega_\chi(z)\varepsilon_\iota(\chi). \ \square$$

We may now describe how we obtain information about the $\varepsilon_\iota(\pi)$'s of G from the Frobenius-Schur indicators of $G^{\iota,z}$.

Proposition 2.2. *Suppose there is a z in the center of G such that for every irreducible ϕ of $G^{\iota,z}$ with central character ω_ϕ we have $\varepsilon(\phi) = \omega_\phi(z)$. Suppose further that $\hat{\pi} \cong {}^\iota\pi$ for every irreducible π of G. Then $\varepsilon_\iota(\pi) = 1$ for every irreducible π of G.*

Proof. Take any $\pi \in \mathrm{Irr}(G)$, and first assume that π^+ is reducible. Then $\pi^+ = \phi_1 + \phi_2$ for ϕ_1, ϕ_2 irreducible representations of $G^{\iota,z}$. From Lemma 2.3, we have

$$\frac{1}{|G^{\iota,z}|} \sum_{g \in G^{\iota,z}} \chi^+(g^2) = \varepsilon(\pi) + \omega_\pi(z)\varepsilon_\iota(\pi).$$

Since π^+ is reducible, we have $\chi^+ = \psi_1 + \psi_2$, where χ^+ is the character of π^+, and ψ_1, ψ_2 are the characters of ϕ_1, ϕ_2, respectively, and so

$$\frac{1}{|G^{\iota,z}|} \sum_{g \in G^{\iota,z}} \chi^+(g^2) = \frac{1}{|G^{\iota,z}|} \sum_{g \in G^{\iota,z}} \psi_1(g^2) + \frac{1}{|G^{\iota,z}|} \sum_{g \in G^{\iota,z}} \psi_2(g^2) = \varepsilon(\phi_1) + \varepsilon(\phi_2).$$

So now $\varepsilon(\phi_1) + \varepsilon(\phi_2) = \varepsilon(\pi) + \omega_\pi(z)\varepsilon_\iota(\pi)$. Now, $\omega_\pi(z) = \omega_{\phi_1}(z) = \omega_{\phi_2}(z)$, since ϕ_1 and ϕ_2 are both constituents of the representation induced from π. So now $\varepsilon(\phi_1) = \varepsilon(\phi_2) = \omega_\pi(z)$. If $\omega_\pi(z) = 1$, then $\varepsilon(\pi) = 1$ and $\varepsilon_\iota(\pi) = 1$. If $\omega_\pi(z) = -1$, then $\varepsilon(\pi) = -1$ and $\varepsilon_\iota(\pi) = 1$.

Now take π to be an irreducible representation of G such that π^+ is irreducible. By Lemma 2.2, $\pi \not\cong {}^\iota\pi$. We have assumed that $\hat{\pi} \cong {}^\iota\pi$, and so $\hat{\pi} \not\cong \pi$, which means $\varepsilon(\pi) = 0$. Since π^+ is irreducible, Lemma 2.3 says that $\varepsilon(\pi^+) = \varepsilon(\pi) + \omega_\pi(z)\varepsilon_\iota(\pi) = \omega_\pi(z)\varepsilon_\iota(\pi)$. Since π^+ is the representation induced from π, we must have $\omega_{\pi^+}(z) = \omega_\pi(z)$. Since $\varepsilon(\pi^+) = \omega_{\pi^+}(z)$, we must have $\varepsilon_\iota(\pi) = 1$. □

We note that Proposition 2.2 can also be proven using the meanings of Frobenius-Schur indicators in terms of the bilinear forms in Equations (2) and (3).

When $z = 1$, denote $G^{\iota,z} = G^+$. In Theorem 1.1, Gow proves that for $G = GL(n, \mathbb{F}_q)$ and ι the inverse-transpose automorphism, every irreducible π of G^+ satisfies $\varepsilon(\pi) = 1$. Since $z = 1$ in this case, this is the same as proving that $\varepsilon(\pi) = \omega_\pi(z)$ always holds. Since every element of $GL(n, \mathbb{F}_q)$ is conjugate to its transpose, it is also true that $\hat{\pi} \cong {}^\iota\pi$ for every irreducible π of G. So the conclusions of Proposition 2.2 follow, and Proposition 2.1 may be applied to see Klyachko and Gow's result of Corollary 1.1. It should also be noted that if $z = 1$ for Proposition 2.2, there is also the consequence that $\varepsilon(\pi) \geq 0$ for every irreducible π of G. So a consequence of Theorem 1.1 of Gow is that $\varepsilon(\pi) \geq 0$ for every irreducible π of $GL(n, \mathbb{F}_q)$. Prasad [18, Theorem 4] also proves this fact using parabolic induction.

3. Conjugacy Properties in Symplectic and Similitude Groups

Let F be a field such that $\mathrm{char}(F) \neq 2$, V a $2n$-dimensional F-vector space, and let $\langle \cdot, \cdot \rangle : V \times V \to F$ be a nondegenerate skew-symmetric bilinear form. The *group of similitudes of* $\langle \cdot, \cdot \rangle$, (or *general symplectic group*) is defined as $GSp(2n, F) = \{g \in GL(2n, F) : \langle gv, gw \rangle = \mu(g)\langle v, w \rangle$ for some $\mu(g) \in F^\times$ for all $v, w \in V\}$. The function $\mu : GSp(2n, F) \to F^\times$ is a multiplicative character called the *similitude character*. Then the *symplectic group* $Sp(2n, F)$ is the subgroup of $GSp(2n, F)$ which is the kernel of μ, leaving the inner product invariant. We will also write $GSp(V) = GSp(2n, F)$ and $Sp(V) = Sp(2n, F)$.

The following proposition is a generalization of a result of Gow [10, Lemma 1]. A proof due to Bump and Ginzburg appears in [19, Proposition 4].

Proposition 3.1. *Let V be an F-vector space such that* $\mathrm{char}(F) \neq 2$, *equipped with a nondegenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$.*

(i) If $-\beta \in F$ is a square in F, there exists a unique conjugacy class of $GSp(V)$ whose elements g satisfy $g^2 = -\beta I$, $\mu(g) = \beta$.

(ii) Suppose that $-\beta \in F$ is not a square in F, and let K be a quadratic extension of F containing the square roots of $-\beta$. Let $\varphi : \lambda \mapsto \overline{\lambda}$ be the nontrivial element of $Gal(K/F)$. If the norm map $N : K \to F$, $N(\lambda) = \lambda\overline{\lambda}$ is surjective, then there exists a unique conjugacy class of $GSp(V)$ whose elements g satisfy $g^2 = -\beta I$, $\mu(g) = \beta$.

Now let the field F be the finite field $\mathbb{F}_q$ of odd characteristic. We consider $V = \mathbb{F}_q^{2n}$ with the standard basis, and let $Sp(2n, \mathbb{F}_q)$ be the transformations on V leaving $\langle v, w \rangle = {}^t v J w$ invariant, where $J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$.

Consider the Frobenius automorphism φ of $GL(n, \mathbb{F}_{q^2})$, which raises each entry of an element g to the power q, and write ${}^\varphi g = g^{(q)}$. Define the *unitary group* over $\mathbb{F}_{q^2}$, $U(n, \mathbb{F}_{q^2})$, to be the subgroup of elements g of $GL(n, \mathbb{F}_{q^2})$ that satisfy $g\, {}^t g^{(q)} = I$.

We have the following Proposition, whose proof is adapted from unpublished notes of Bump and Ginzburg, which gives more information about the conjugacy class described in Proposition 3.1.

Proposition 3.2. *Let q be odd. For each $\beta \in \mathbb{F}_q^\times$, there is a unique conjugacy class in $GSp(2n, \mathbb{F}_q)$ of elements g satisfying $g^2 = -\mu(g)I$ and $\mu(g) = \beta$. The centralizer of this conjugacy class contains a subgroup of index $q - 1$ isomorphic to*

$$\begin{cases} GL(n, \mathbb{F}_q) & \text{if } -\beta \text{ is a square} \\ U(n, \mathbb{F}_{q^2}) & \text{if } -\beta \text{ is not a square.} \end{cases}$$

Proof. First, since the norm map from $\mathbb{F}_{q^2}$ down to $\mathbb{F}_q$ is surjective, then by Proposition 3.1 the conjugacy class described is unique. For $\beta \in \mathbb{F}_q^\times$, the element

$$g = \begin{pmatrix} & I_n \\ -\beta I_n & \end{pmatrix}$$

of $GSp(2n, \mathbb{F}_q)$ satisfies $\mu(g) = \beta$ and $g^2 = -\beta I$. It is a direct computation that the centralizer of g in $GSp(2n, \mathbb{F}_q)$ is given by

$$C(g) = \left\{ h = \begin{pmatrix} A & B \\ -\beta B & A \end{pmatrix} \mid {}^tAB = {}^tBA, {}^tAA + \beta\, {}^tBB = \mu(h)I \right\}.$$

We note that $C(g)$ contains elements of every similitude, for if $\lambda \in \mathbb{F}_q^\times$, then the element $h \in C(g)$ with $B = I_n$, and

$$A = \begin{pmatrix} aI & bI \\ bI & -aI \end{pmatrix},$$

where $a^2+b^2 = \lambda$, satisfies $\mu(h) = \lambda$. So the subgroup of symplectic elements in $C(g)$, or the set of elements of $\ker(\mu)$ in $C(g)$, call it $C_1(g)$, is a normal subgroup in $C(g)$ of index $q-1$.

Now let γ be a square root of $-\beta$, so if $-\beta$ is not a square in $\mathbb{F}_q$, then $\gamma \in \mathbb{F}_{q^2}$. Define a map on $C_1(g)$ by

$$F : \begin{pmatrix} A & B \\ -\beta B & A \end{pmatrix} \mapsto A + \gamma B.$$

Then F is a multiplicative homomorphism, and the image satisfies

$${}^t(A + \gamma B)(A - \gamma B) = I,$$

so if $-\beta$ is a square, the image of F is in $GL(n, \mathbb{F}_q)$, and if $-\beta$ is not a square, then the image is in $GL(n, \mathbb{F}_{q^2})$. Letting $C = A + \gamma B$, we have ${}^tC^{-1} = A - \gamma B$, and so given C in the image of F, we find unique A and B mapping to C,

$$A = \frac{1}{2}(C + {}^tC^{-1}), \quad B = \frac{1}{2\gamma}(C - {}^tC^{-1}),$$

so that F is injective.

If $-\beta$ is a square in $\mathbb{F}_q^\times$, then for any $C \in GL(n, \mathbb{F}_q)$, we may choose A and B over $\mathbb{F}_q$ as above, and so F is surjective. It follows that $C_1(g)$ is isomorphic to $GL(n, \mathbb{F}_q)$ in this case.

If $-\beta$ is not a square, then ${}^\varphi C = C^{(q)} = {}^tC^{-1}$, since ${}^\varphi\gamma = \gamma^q = -\gamma$, so that the image of F is contained in $U(n, \mathbb{F}_{q^2})$. In that case, the choices for A and B above are stable under the Frobenius, and so defined over $\mathbb{F}_q$. It follows that F surjects onto $U(n, \mathbb{F}_{q^2})$, and so in this case $C_1(g)$ is isomorphic to $U(n, \mathbb{F}_{q^2})$. $\square$

We call $g \in GSp(2n, F)$ a *skew-symplectic involution* if $\mu(g) = -1$ and $g^2 = I$. Wonenburger proved the following in [21].

Theorem 3.1. *Let $G = Sp(2n, F)$ where* $\mathrm{char}(F) \neq 2$. *Then every element of $g \in G$ may be written $g = h_1 h_2$, where h_1 and h_2 are skew-symplectic involutions.*

Going back to the case that $F = \mathbb{F}_q$ for q odd, let ι be the automorphism of $Sp(2n, \mathbb{F}_q)$ that conjugates elements by the skew-symplectic involution $\begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}$. We immediately apply Wonenburger's result to the case at hand.

Proposition 3.3. *Let $G = Sp(2n, \mathbb{F}_q)$ with q odd and ι defined as above. For every $g \in G$, there is an element $s \in G$ such that $s^{-1}gs = {}^{\iota}g^{-1}$, and such that ${}^{\iota}s^{-1} = s$. In particular, we have g^{-1} is conjugate to ${}^{\iota}g$ in G, and every irreducible character χ of G satisfies $\varepsilon_{\iota}(\chi) = \pm 1$.*

Proof. The first statement implies the second since if for every $g \in G$, we have ${}^{\iota}g$ is conjugate to g^{-1}, we have $\chi({}^{\iota}g) = \overline{\chi(g)}$. From Theorem 3.1, we know that there exists an element $h \in GSp(2n, \mathbb{F}_q)$ with $\mu(h) = -1$, $h^2 = I$, and such that $h^{-1}gh = g^{-1}$. Now let $t = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}$, which is itself a skew-symplectic involution. Conjugating both sides by t, we have $(th)^{-1}g(th) = {}^{\iota}g^{-1}$. Since $\mu(t) = \mu(h) = -1$, we have $\mu(th) = 1$, and so in fact $th \in G$. Letting $s = th$, we have ${}^{\iota}s^{-1} = {}^{\iota}(th)^{-1} = t(ht)t = th = s$. □

As a consequence of Proposition 3.3, the automorphism ι must be outer when $q \equiv 3(\mathrm{mod}\ 4)$. Otherwise, since $Sp(2n, \mathbb{F}_q)$ has irreducible complex characters χ such that $\varepsilon(\chi) = 0$ (for example, by [5, Lemma 5.3]), then by Lemma 2.1, we would have $\varepsilon_{\iota}(\chi) = 0$, contradicting the proposition.

The following results from [19, Corollary 1 and Theorem 4] are useful in our situation.

Proposition 3.4. *Let $g \in GSp(2n, F)$, where $\mathrm{char}(F) \neq 2$. Then g is conjugate to $\mu(g)g^{-1}$ by a skew-symplectic involution.*

Proposition 3.5. *Let $G = Sp(2n, \mathbb{F}_q)$, where $q \equiv 3(\mathrm{mod}\ 4)$. Let ι be the order 2 automorphism of G defined by the following conjugation by a skew-symplectic element:*

$$ {}^{\iota}g = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix} g \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}. $$

Define $G^{\iota, -I}$ as the following group containing G as an index 2 subgroup:

$$ G^{\iota,-I} = \langle G, \tau \mid \tau^2 = -I, \tau^{-1}g\tau = {}^{\iota}g \text{ for every } g \in G \rangle. $$

Then every element of $G^{\iota,-I}$ is conjugate to its inverse, and so every complex character of $G^{\iota,-I}$ is real-valued.

For $q \equiv 3(\mathrm{mod}\ 4)$, we now have that $\varepsilon_{\iota}(\chi) = \pm 1$ for any irreducible χ of $G = Sp(2n, \mathbb{F}_q)$, by Proposition 3.3, and that every irreducible of $G^{\iota,-I}$ is real-valued. So to obtain that $\varepsilon_{\iota}(\chi) = 1$ for every irreducible χ of G, by Proposition 2.2, we need to show that $\varepsilon(\psi) = \omega_{\psi}(-I)$ for every irreducible ψ of $G^{\iota,-I}$. Because of Gow's Theorem 1.2, though, we can already say the following.

Proposition 3.6. *Let $G = Sp(2n, \mathbb{F}_q)$ with $q \equiv 3 (mod\ 4)$, and let ι be the order 2 automorphism of G defined as before.*

(i) If χ is an irreducible real-valued character of G, then $\varepsilon_\iota(\chi) = 1$.

(ii) $\varepsilon_\iota(\chi) = 1$ for every irreducible χ of G if and only if $\varepsilon(\psi) = \omega_\psi(-I)$ for every irreducible ψ of $G^{\iota,-I}$.

Proof. For (i), Gow's Theorem 1.2 states that $\omega_\chi(-I) = \varepsilon(\chi)$ for real-valued χ of G. Since χ is real-valued and ${}^\iota\chi = \overline{\chi}$ by Proposition 3.3, then by Lemma 2.2, we have $\chi^{G^{\iota,-I}} = \psi_1 + \psi_2$ where ψ_1 and ψ_2 are irreducible characters of $G^{\iota,-I}$ which are extensions of χ. Also ψ_1 and ψ_2 are real-valued by Proposition 3.5. From Lemma 2.3,

$$\varepsilon(\psi_1) + \varepsilon(\psi_2) = \varepsilon(\chi) + \omega_\chi(-I)\varepsilon_\iota(\chi).$$

Now, $\omega_\chi(-I) = \varepsilon(\chi)$ and $\varepsilon_\iota(\chi) = \pm 1$. If $\varepsilon(\chi) = -1$, then we must have $\varepsilon(\psi_1) = \varepsilon(\psi_2) = -1$ since they are both extensions of χ and real-valued. Then since $\omega_\chi(-I) = \varepsilon(\chi)$, we must have $\varepsilon_\iota(\chi) = 1$. If $\varepsilon(\chi) = 1$, then at least one of $\varepsilon(\psi_1)$ or $\varepsilon(\psi_2)$ must be 1 by the equation above, but then they must both be 1 since $\chi^{G^{\iota,-I}}$ is a real representation, and real subrepresentations must have real complements. Again this implies $\varepsilon_\iota(\chi) = 1$.

For (ii), the "if" part is exactly Proposition 2.2. For the "only if", we have the analysis above when χ is real-valued and $\chi^{G^{\iota,-I}} = \psi_1 + \psi_2$, and this case follows from the fact that $\omega_\chi(-I) = \omega_{\psi_1}(-I) = \omega_{\psi_2}(-I)$. When χ is not real-valued, we have $\chi^{G^{\iota,-I}} = \psi$ is irreducible from Lemma 2.2, and $\varepsilon(\psi) = \omega_\chi(-I)\varepsilon_\iota(\chi)$ from Lemma 2.3, where $\omega_\chi(-I) = \omega_\psi(-I)$. □

4. Brauer-Witt-Berman Induction

The following proposition is the main tool to be used in an induction argument for the main theorem, in the same way it is used in [8, 9]. Part (i) is a result coming from the Witt-Berman generalization of Brauer's induction theorem. An $\mathbb{R}$-elementary subgroup at 2, of a finite group G, is a subgroup which is a semidirect product, $\langle a \rangle B$, such that a has odd order and B is a 2-group such that for every $b \in B$, we have $b^{-1}ab = a$ or a^{-1}.

Proposition 4.1. *Let χ be a real-valued irreducible complex character of a finite group G. Then:*
(i) There exists an $\mathbb{R}$-elementary subgroup at 2, H of G, and a real-valued irreducible character ψ of H such that $\langle \chi, \psi^G \rangle$ is an odd integer, and $\varepsilon(\chi) = \varepsilon(\psi)$.
(ii) If $H = \langle a \rangle B$ and ψ are as in (i), then either H may be taken so that $a = 1$, or ψ may be taken so that $\langle a \rangle \not\subset \ker(\psi)$.
(iii) For any subgroup M of G that contains H, there is an irreducible real-valued character θ of M such that $\langle \chi, \theta^G \rangle$ is odd, and $\varepsilon(\chi) = \varepsilon(\psi) = \varepsilon(\theta)$.
(iv) H can be taken to be either a Sylow 2-subgroup of G, or $\langle a \rangle B$, where a is a real element of G with odd order and B is a Sylow 2-subgroup of $N_{\mathbb{R}}(a) = \{x \in G \mid x^{-1}ax = a \text{ or } a^{-1}\}$. That is, we may assume H is a maximal $\mathbb{R}$-elementary subgroup at 2.

Proof. (i): The fact that there is a subgroup H with a real-valued ψ such that $\langle \chi, \psi^G \rangle$ is odd is a special case of the more general statement coming from the Witt-Berman Theorem, as proven in [3, Lemma 70.25] and [4, 15.12]. The fact that $\varepsilon(\chi) = \varepsilon(\psi)$ may be concluded by using divisibility properties of the Schur index. See, for example, [14, Corollary 10.2 (c)].

(ii): Let $H = \langle a \rangle B$ be the $\mathbb{R}$-elementary subgroup at 2 of G from part (i). Suppose that $a \neq 1$ and $\langle a \rangle \subset \ker(\psi)$. Then $\eta = \psi|_B$ is an irreducible real-valued character of the 2-group B. If $\langle \eta^G, \chi \rangle$ is odd, then we are done by just replacing H by B and ψ by η. So suppose $\langle \eta^G, \chi \rangle$ is even. Consider the induced character η^H. Since $\eta = \psi|_B$, then by Frobenius reciprocity we have $\langle \eta^H, \psi \rangle = \langle \eta, \psi|_B \rangle = 1$. Since η is real-valued, η^H is also, and the non-real-valued constituents of η^H will have the same multiplicity as their conjugates. So writing η^H as a sum of irreducibles of H, we have

$$\eta^H = \psi + \sum_i a_i \theta_i + \sum_j b_j (\xi_j + \overline{\xi_j}),$$

where ψ is different from the θ_i's and ξ_j's, and the θ_i are real-valued while the ξ_j are not. Now induce both sides to G, and take the inner product with χ. Since we are assuming that $\langle \eta^G, \chi \rangle$ is even, while $\langle \psi^G, \chi \rangle$ is odd, we must have that $\langle \theta_i^G, \chi \rangle$ is odd for at least one of the θ_i's, let θ be one of them. We must finally show that $\langle a \rangle \not\subset \ker(\theta)$. If not, then $\theta|_B$ is an irreducible of B. Since $\psi \neq \theta$, and $\langle a \rangle \subset \ker(\psi)$, we have $\theta|_B \neq \psi|_B$. But then we have

$$\langle \eta^H, \theta \rangle = \langle \eta, \theta|_B \rangle = \langle \psi|_B, \theta|_B \rangle = 0,$$

a contradiction since θ was taken as a constituent of η^H. So now $\langle a \rangle \not\subset \ker(\theta)$, and we may replace ψ by θ, since $\langle \theta^G, \chi \rangle$ is odd.

(iii): Take the ψ obtained from part (i) of H, which is real-valued. Write ψ^M as a sum of irreducibles of M, $\psi^M = \sum_i a_i \theta_i$, where, since ψ^M is real-valued, each θ_i appears in the sum with the same multiplicity as its conjugate. Since $(\psi^M)^G = \psi^G$, we have, where $\mathrm{T}(\theta_j) = \mathrm{Tr}_{\mathbb{R}(\theta_j)/\mathbb{R}}(\theta_j)$,

$$\langle \psi^G, \chi \rangle = \sum_j b_j \langle \mathrm{T}(\theta_j)^G, \chi \rangle = \sum_j b_j [\mathbb{R}(\theta_j) : \mathbb{R}] \langle \theta_j^G, \chi \rangle.$$

But $\langle \psi^G, \chi \rangle$ is odd, and so there must be a $\theta_j = \theta$ of M such that $\langle \theta^G, \chi \rangle$ is odd and θ is real-valued. Then $\varepsilon(\theta) = \varepsilon(\chi) = \varepsilon(\psi)$ from divisibility properties of the Schur index.

(iv): First, a may be taken to be real if it is not 1, because otherwise the $\mathbb{R}$-elementary subgroup at 2, H, is of the form $\langle a \rangle \times B$. Then a real-valued irreducible ψ of this would have to be the product of a real-valued irreducible of $\langle a \rangle$ with a real-valued irreducible of B. But since a has odd order, the only irreducible real-valued character it has is the trivial character, which would mean $\langle a \rangle$ would be in $\ker(\psi)$, which may be avoided by part (ii). We may then take H to be a maximal $\mathbb{R}$-elementary subgroup at 2 by applying part (iii). $\square$

The idea in applying Proposition 4.1 is to either calculate directly the Frobenius-Schur indicators of the characters of maximal $\mathbb{R}$-elementary subgroups at 2 of a group, or to embed a maximal $\mathbb{R}$-elementary subgroup at 2 in a subgroup that we can handle more easily. Then through Proposition 4.1(i) and (iii), we may find the Frobenius-Schur indicators of the real-valued characters of the whole group. We are interested in doing this for the group $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, $q \equiv 3(\text{mod } 4)$, all of whose characters are real-valued, by Proposition 3.5. For the rest of this section we assume that $q \equiv 3(\text{mod } 4)$.

Thus the task at hand is to show that the required irreducibles ψ of a maximal $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, satisfy $\varepsilon(\psi) = \omega_\psi(-I)$, or to embed the $\mathbb{R}$-elementary subgroup at 2 in another subgroup that satisfies this. We analyze the $\mathbb{R}$-elementary subgroups at 2 of the group $Sp(2n, \mathbb{F}_q)^{\iota,-I}$ in the next section, and we will embed many of these subgroups in two types of subgroups that are described in the remainder of this section.

In the end, the proof will be by induction on n, and we now explain the base case $n = 1$. From the character table of $SL(2, \mathbb{F}_q)$, q odd, we compute that the sum of the degrees of the irreducible characters is $q^2 + q$. We may also count that the number of matrices in $SL(2, \mathbb{F}_q)$ which satisfy $g\ {}^{\iota}g = I$, or ${}^{\iota}g = g^{-1}$, is equal to $q^2 + q$. So by Proposition 2.1(ii), we have that every character χ of $SL(2, \mathbb{F}_q)$ satisfies $\varepsilon_\iota(\chi) = 1$. Now by Proposition 3.6(ii), this implies that $\varepsilon(\psi) = \omega_\psi(-I)$ for every irreducible ψ of $SL(2, \mathbb{F}_q)^{\iota,-I}$ when $q \equiv 3(\text{mod } 4)$. The base case is proven, and now the induction hypothesis is that this holds true for all irreducible characters of $Sp(2m, \mathbb{F}_q)^{\iota,-I}$, $q \equiv 3(\text{mod } 4)$, for all $m < n$.

Assuming the induction hypothesis, we have a family of subgroups of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$ whose Frobenius-Schur indicators we may calculate. If $m_1 + m_2 = n$, we embed $Sp(2m_1, \mathbb{F}_q) \times Sp(2m_2, \mathbb{F}_q)$ in $Sp(2n, \mathbb{F}_q)$ orthogonally. Then the action of ι on each of the smaller symplectic groups in the product acts the same as ι on the large symplectic group.

Proposition 4.2. *Let $m_1 + m_2 + \ldots + m_r = n$, with m_i positive integers. Let $M = Sp(2m_1, \mathbb{F}_q) \times \cdots \times Sp(2m_r, \mathbb{F}_q)$. Then assuming the induction hypothesis stated above, every irreducible character χ of $M^{\iota,-I} \cong \langle M, \tau \rangle$ satisfies $\varepsilon(\chi) = \omega_\chi(-I)$.*

Proof. Any irreducible character ψ of M is of the form $\psi_1\psi_2 \cdots \psi_r$, where ψ_i is an irreducible character of $M_i = Sp(2m_i, \mathbb{F}_q)$. From the induction hypothesis, we know that $\varepsilon_\iota(\psi_i) = 1$ for any ψ_i. Since ι acting on each factor in the product acts like ι on M, we have, using the formula for $\varepsilon_\iota(\psi)$,

$$\varepsilon_\iota(\psi) = \varepsilon_\iota(\psi_1) \cdots \varepsilon_\iota(\psi_r) = 1.$$

Now, every irreducible χ of $M^{\iota,-I}$ is either isomorphic to ψ induced to $M^{\iota,-I}$, which we write ψ^+, for some irreducible ψ of M, or is an extension of an irreducible ψ, that is, $\psi^+ = \chi + \chi'$ for some other irreducible χ'.

In the first case, Lemma 2.2 says that ψ^+ is irreducible when $\varepsilon(\psi) = 0$, so by Lemma 2.3, $\varepsilon(\chi) = \omega_\psi(-I)\varepsilon_\iota(\psi) = \omega_\psi(-I)$, since $\varepsilon_\iota(\psi) = 1$. But now $\omega_\psi(-I) = \omega_\chi(-I)$, so we have $\varepsilon(\chi) = \omega_\chi(-I)$. In the other case we have $\psi^+ = \chi + \chi'$. By Lemma 2.3, $\varepsilon(\chi) + \varepsilon(\chi') = \varepsilon(\psi) + \omega_\psi(-I)\varepsilon_\iota(\psi)$. But $\varepsilon_\iota(\psi) = 1$ and $\varepsilon(\psi) = \omega_\psi(-I)$ by Gow's Theorem 1.2, and $\omega_\psi(-I) = \omega_\chi(-I)$, so $\varepsilon(\chi) = \omega_\chi(-I)$. □

Recall that the wreath product of a group H with $\mathbb{Z}/2\mathbb{Z}$ is the group

$$\langle \sigma, \, (h_1, h_2) \in H \times H \mid \sigma^2 = 1, \, \sigma(h_1, h_2)\sigma = (h_2, h_1) \rangle.$$

If H is a subgroup of $Sp(2n, \mathbb{F}_q)$, then we embed $H \times H$ in $Sp(4n, \mathbb{F}_q)$ orthogonally, and we may take

$$\sigma = \begin{pmatrix} & I_n & & \\ I_n & & & \\ & & & I_n \\ & & I_n & \end{pmatrix}.$$

This embeds the wreath product inside of $Sp(4n, \mathbb{F}_q)$. Note that if H is fixed by ι, then since σ is fixed by ι, we have the wreath product of H with $\mathbb{Z}/2\mathbb{Z}$ is fixed by ι. We give the following result, whose proof is very similar to the proof of Gow [8, Lemma 4], except the twist by ι is added.

Proposition 4.3. *Let H be a subgroup of $Sp(2n, \mathbb{F}_q)$ fixed by ι and let K the wreath product of H with $\mathbb{Z}/2\mathbb{Z}$, which is thus a subgroup of $Sp(4n, \mathbb{F}_q)$ fixed by ι. If $\varepsilon_\iota(\psi) = 1$ for every irreducible ψ of H, then $\varepsilon_\iota(\chi) = 1$ for every irreducible χ of K.*

Proof. Any irreducible of K is either induced or extended from an irreducible of $H \times H$, since this is a subgroup of index 2. Any irreducible of $H \times H$ is of the form $\psi_i\psi_j$, where ψ_i, ψ_j are irreducibles of H. Then $\psi_i\psi_j$ induced to H is irreducible if and only if $i \neq j$, and in this case $\psi_i\psi_j$ and $\psi_j\psi_i$ induce to the same character. If $i = j$, then $\psi_i\psi_j$ may be extended to K in two different ways. This means that the sum of the degrees of the characters of K is

$$2\sum_{i<j} \psi_i(1)\psi_j(1) + 2\sum_i \psi_i(1)^2 = \big(\sum_i \psi_i(1)\big)^2 + \sum_i \psi_i(1)^2,$$

where $\{\psi_i\}_i$ is the set of irreducible characters of H. We know that $\varepsilon_\iota(\psi_i) = 1$ for every ψ_i of H. By Proposition 2.1(ii), we have

$$\sum_i \psi_i(1) = \big|\{h \in H \mid {}^\iota h = h^{-1}\}\big|,$$

and let us call this quantity c. Since $\sum_i \psi_i(1)^2 = |H|$, we have that the sum of the degrees of the irreducible characters of K is equal to $c^2 + |H|$. We now show that this is the same number of elements $g \in K$ such that ${}^\iota g = g^{-1}$.

There are 2 cosets of $N = H \times H$ in K, and call the order 2 element that acts on $H \times H$ by transposition σ. An element $(h_1, h_2) \in N$ has image

$({}^{\iota}h_1, {}^{\iota}h_2)$ under the automorphism ι of K. Since $(h_1, h_2)^{-1} = (h_1^{-1}, h_2^{-1})$, in order for $(h_1, h_2) = g \in N$ to satisfy ${}^{\iota}g = g^{-1}$, we must have ${}^{\iota}h_1 = h_1^{-1}$ and ${}^{\iota}h_2 = h_2^{-1}$. The number of such elements in this coset is c^2.

Since ${}^{\iota}\sigma = \sigma = \sigma^{-1}$, an element $\sigma(h_1, h_2) \in \sigma N$ satisfies ${}^{\iota}(\sigma(h_1, h_2)) = (\sigma(h_1, h_2))^{-1}$ when $\sigma({}^{\iota}h_1, {}^{\iota}h_2) = (h_1^{-1}, h_2^{-1})\sigma$. That is, when $({}^{\iota}h_2, {}^{\iota}h_1) = (h_1^{-1}, h_2^{-1})$, or just when ${}^{\iota}h_1 = h_2^{-1}$. So we may choose h_1 to be any element of H, and h_2 is determined. So the number of elements in the coset σN satisfying ${}^{\iota}g = g^{-1}$ is $|H|$. Now the total number of such elements in K is $c^2 + |H|$, which is also the sum of the degrees of the irreducible characters. So by Proposition 2.1(ii), every irreducible χ of K satisfies $\varepsilon_{\iota}(\chi) = 1$. □

We need to apply the following form of a result of Gow [9, Lemma 2.4].

Lemma 4.1. *Let H be a finite group whose center contains an element z of order 2. Suppose that every real-valued irreducible character χ of H satisfies $\varepsilon(\chi) = \omega_{\chi}(z)$. Let K be the wreath product of H with $\mathbb{Z}/2\mathbb{Z}$. Then every real-valued irreducible θ of K satisfies $\varepsilon(\theta) = \omega_{\theta}(z)$.*

Finally, we have the following proposition for calculating Frobenius-Schur indicators of wreath products of the symplectic group.

Proposition 4.4. *Suppose that $H = Sp(2m, \mathbb{F}_q)$ satisfies the induction hypothesis that $\varepsilon_{\iota}(\chi) = 1$ for every irreducible χ of H. Let K be the wreath product of H with $\mathbb{Z}/2\mathbb{Z}$, viewed as a subgroup of $Sp(4m, \mathbb{F}_q)$. Then every real-valued irreducible ψ of $K^{\iota,-I}$ satisfies $\varepsilon(\psi) = \omega_{\psi}(-I)$*

Proof. By directly applying Proposition 4.3, since K is fixed by ι, we have that $\varepsilon_{\iota}(\theta) = 1$ for every irreducible θ of K. Gow's Theorem 1.2 says that every real-valued irreducible χ of $Sp(2m, \mathbb{F}_q)$ satisfies $\varepsilon(\chi) = \omega_{\chi}(-I)$. So by applying Lemma 4.1, every real-valued irreducible θ of K satisfies $\varepsilon(\theta) = \omega_{\theta}(-I)$. Every irreducible ψ of $K^{\iota,-I}$ is either induced or extended from an irreducible θ of K. Then by Lemma 2.2, ψ is induced from an irreducible θ of K if and only if $\varepsilon(\theta) = 0$, since $\varepsilon_{\iota}(\theta) = 1$. Then $\varepsilon(\psi) = \omega_{\theta}(-I)\varepsilon_{\iota}(\theta)$ by Lemma 2.3. So then $\varepsilon(\psi) = \omega_{\theta}(-I) = \omega_{\psi}(-I)$. If ψ is real-valued and extended from θ of K, then $\theta^{K^{\iota,-I}} = \psi + \psi'$ for another irreducible ψ' of $K^{\iota,-I}$. By Lemma 2.3 we have $\varepsilon(\psi) + \varepsilon(\psi') = \varepsilon(\theta) + \omega_{\theta}(-I)$, since $\varepsilon_{\iota}(\theta) = 1$. Since θ is real-valued in this case, we have $\varepsilon(\theta) = \omega_{\theta}(-I)$. So then we must have $\varepsilon(\psi) = \omega_{\theta}(-I) = \omega_{\psi}(-I)$. □

5. Maximal $\mathbb{R}$-elementary Subgroups at 2

For all of this section we again assume that $q \equiv 3 (\mathrm{mod}\ 4)$. Our application of Proposition 4.1 is that in order to calculate the Frobenius-Schur indicators of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, we need to calculate Frobenius-Schur indicators of maximal $\mathbb{R}$-elementary subgroups at 2 of the form $N = \langle a \rangle B$, where a is a real element of odd order in $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, and B is a Sylow 2-subgroup of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, $N_{\mathbb{R}}(a) = \{x \in Sp(2n, \mathbb{F}_q)^{\iota,-I} \,|\, x^{-1}ax = a$

or $a^{-1}\}$. Call $N = \langle a \rangle B$ the maximal $\mathbb{R}$-elementary subgroup at 2 associated with a. Choosing a different Sylow 2-subgroup of $N_{\mathbb{R}}(a)$ yields an isomorphic $\mathbb{R}$-elementary subgroup at 2, and also choosing a conjugate of a yields an isomorphic $\mathbb{R}$-elementary subgroup at 2. Since a is of odd order, it is an element of the subgroup $Sp(2n, \mathbb{F}_q)$, rather than the other coset, but by Proposition 3.5, every element of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ is real, and so a could be any element of odd order in $Sp(2n, \mathbb{F}_q)$.

In [9], Gow calculated Frobenius-Schur indicators of $Sp(2n, \mathbb{F}_q)$, which he did by analyzing maximal $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)$ of the form $N' = \langle a \rangle B'$, where a is a real element of $Sp(2n, \mathbb{F}_q)$ of odd order, and B' is a Sylow 2-subgroup of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$. Throughout this section, $N' = \langle a \rangle B'$ will always be a maximal $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)$, while $N = \langle a \rangle B$ will be that of the group $Sp(2n, \mathbb{F}_q)^{\iota, -I}$. We may take advantage of Gow's analysis of $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)$ through the following three lemmas. The first follows from an application of the orbit-stabilizer lemma.

Lemma 5.1. *Let $a \in Sp(2n, \mathbb{F}_q)$ have odd order. The maximal $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ associated with a, $N = \langle a \rangle B$, contains as an index 2 subgroup the maximal $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)$ associated with a, $N' = \langle a \rangle B'$. In particular, if s is the element from Proposition 3.3 such that $s^{-1} a s = {}^{\iota} a^{-1}$, then $N \cong \langle N', s\tau \rangle$, where $N' = \langle a \rangle B'$ and B' is a Sylow 2-subgroup of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ fixed by $s\tau$ under conjugation.*

In [9], Gow shows that many $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)$ may be taken to be contained in products or wreath products of smaller symplectic groups. In the next lemma, we apply Lemma 5.1 to show that we may also take the associated $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ to be contained in corresponding subgroups.

Lemma 5.2. *Let $a \in Sp(2n, \mathbb{F}_q)$ have odd order. Suppose that the maximal $\mathbb{R}$-elementary subgroup at 2, $N' = \langle a \rangle B'$ in $Sp(2n, \mathbb{F}_q)$ associated with a, is such that either*

(i) we may take a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ to be contained in a subgroup of the form $M = Sp(2n_1, \mathbb{F}_q) \times \cdots \times Sp(2n_t, \mathbb{F}_q) \subset Sp(2n, \mathbb{F}_q)$, so that N' is contained in M, or

(ii) we may take a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ to be contained in a subgroup $K \subset Sp(2n, \mathbb{F}_q)$, where K is the wreath product of $Sp(n, \mathbb{F}_q)$ with $\mathbb{Z}/2\mathbb{Z}$, so that N' is contained in K.

Then, if $N = \langle a \rangle B$ is the maximal $\mathbb{R}$-elementary subgroup at 2 associated with a in $Sp(2n, \mathbb{F}_q)^{\iota, -I}$, we have, for the conditions above respectively, that

(i) N may be taken to be in a subgroup of the form $M^{\iota, -I} = \langle M, \tau \rangle \subset Sp(2n, \mathbb{F}_q)^{\iota, -I}$, or

(ii) N may be taken to be in a subgroup of the form $K^{\iota, -I} = \langle K, \tau \rangle \subset Sp(2n, \mathbb{F}_q)^{\iota, -I}$.

Proof. In (i), since $a \in M$, then we may take a as $a = (a_1, \ldots, a_t) \in M$, where $a_i \in Sp(2n_i, \mathbb{F}_q)$. For each a_i, take the element $s_i \in Sp(2n_i, \mathbb{F}_q)$ that exists from Proposition 3.3, so then we may take $s = (s_1, \ldots, s_t) \in M$. The $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ is contained in M, and conjugation by $s\tau$ is an order 2 automorphism. Since there are an odd number of Sylow 2-subgroups, we may find a Sylow 2-subgroup fixed by conjugation by $s\tau$ contained in M, call it B'. Then from Lemma 5.1, $N \cong \langle N', s\tau \rangle$, where $N' = \langle a \rangle B'$. Since $N' \subset M$ and $s \in M$, we have $N = \langle N', s\tau \rangle \subset \langle M, \tau \rangle$.

In (ii), if $a \in K$, and since a has odd order, then $a \in Sp(n, \mathbb{F}_q) \times Sp(n, \mathbb{F}_q)$, rather than the other coset in K. As before we may take $s \in Sp(n, \mathbb{F}_q) \times Sp(n, \mathbb{F}_q)$, and a Sylow 2-subgroup B' of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ fixed by conjugation by $s\tau$. Now, $N \cong \langle N', s\tau \rangle$, where $N' = \langle a \rangle B'$, from Lemma 5.1. Then $N' \subset K$ and $s \in K$, so $N = \langle N', s\tau \rangle \subset \langle K, \tau \rangle$. $\square$

In some cases, Gow computes the required Frobenius-Schur indicators of $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)$ directly. The following lemma shows that we may compute the Frobenius-Schur indicators of the associated $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ from Gow's computations, under suitable conditions.

Lemma 5.3. *Let H be a finite group, λ an order 2 automorphism of H, z an element of the center of H, and let*

$$H^{\lambda, z} = \langle H, u \mid u^2 = z, u^{-1} h u = {}^{\lambda} h \text{ for all } h \in H \rangle.$$

(i) Let T be a normal subgroup of H and $H^{\lambda,z}$. Suppose that each irreducible character θ of H satisfying $T \not\subset \ker(\theta)$ is either real-valued and satisfies $\varepsilon(\theta) = \omega_\theta(z)$, or is not real-valued and satisfies $\varepsilon_\lambda(\theta) \neq -1$. Then every real-valued irreducible characters χ of $H^{\lambda,z}$ such that $T \not\subset \ker(\chi)$ satisfies $\varepsilon(\chi) = \omega_\chi(z)$.

(ii) Suppose that each irreducible character of H is either real-valued and satisfies $\varepsilon(\theta) = \omega_\theta(z)$, or is not real-valued and satisfies $\varepsilon_\lambda(\theta) \neq -1$. Then every irreducible real-valued character χ of $H^{\lambda,z}$ satisfies $\varepsilon(\chi) = \omega_\chi(z)$.

Proof. (i): Let χ be an irreducible real-valued character of $H^{\lambda,z}$ such that $T \not\subset \ker(\chi)$. Then χ is either extended or induced from an irreducible character θ of H. Supposing first that χ is extended from an irreducible θ of H, then θ must also be real-valued and satisfy $T \not\subset \ker(\theta)$, and so by assumption θ satisfies $\varepsilon(\theta) = \omega_\theta(z)$. In this case, χ is a constituent of θ induced to $H^{\lambda,z}$, and by Lemma 2.3, we have

$$\varepsilon(\chi) + \varepsilon(\chi') = \varepsilon(\theta) + \omega_\theta(z)\varepsilon_\lambda(\theta),$$

where χ' is another extension of θ, and χ' is real-valued since χ and θ are. If $\varepsilon(\theta) = -1$, then θ cannot extend to a real representation of $H^{\lambda,z}$, and we must have $\varepsilon(\chi) = \varepsilon(\chi') = -1$. Then $\varepsilon(\chi) = \omega_\theta(z) = \omega_\chi(z)$. If $\varepsilon(\theta) = 1$, then θ induced to $H^{\lambda,z}$ is a real representation. But at least one of $\varepsilon(\chi)$ or $\varepsilon(\chi')$ must be 1, since they are both ± 1, and their sum is at least 0. By

Maschke's Theorem, a real subrepresentation of a real representation must have a real complement, so that $\varepsilon(\chi) = 1 = \omega_\chi(z)$.

If χ is induced from an irreducible θ of H, then we must have $T \not\subset \ker(\theta)$, because otherwise $T \subset \ker(\chi)$. If θ is real-valued, then since ${}^\lambda\theta \neq \theta$ by Lemma 2.2, we have $\varepsilon_\lambda(\theta) = 0$. So by Lemma 2.3, and since $\varepsilon(\theta) = \omega_\theta(z)$, we have

$$\varepsilon(\chi) = \varepsilon(\theta) + \omega_\theta(z)\varepsilon_\lambda(\theta) = \varepsilon(\theta) + 0 = \omega_\theta(z) = \omega_\chi(z).$$

If θ is not real-valued, we have $\varepsilon(\theta) = 0$, and by Lemma 2.3,

$$\varepsilon(\chi) = \varepsilon(\theta) + \omega_\theta(z)\varepsilon_\lambda(\theta) = \omega_\theta(z)\varepsilon_\lambda(\theta).$$

Since χ is assumed to be real-valued, then $\varepsilon(\chi) \neq 0$, and so $\varepsilon_\lambda(\theta) \neq 0$. Since $\varepsilon_\lambda(\theta) \neq -1$ by assumption, then $\varepsilon_\lambda(\theta) = 1$ and so $\varepsilon(\chi) = \omega_\theta(z) = \omega_\chi(z)$. Now for every real-valued irreducible χ satisfying $T \not\subset \ker(\chi)$, we have $\varepsilon(\chi) = \omega_\chi(z)$.

(ii): The statement quickly follows from (i) by letting $T = H$. $\square$

Lemma 5.3 will typically be applied in the situation that H is a maximal $\mathbb{R}$-elementary subgroup at 2, N', of $Sp(2n, \mathbb{F}_q)$ associated with an odd order real element a. Then, as in Lemma 5.1, the maximal $\mathbb{R}$-elementary subgroup at 2, N, of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$ associated with a is isomorphic to a group generated by N' and the element $s\tau$ of Proposition 3.3. Then $(s\tau)^2 = -I$, and conjugation by $s\tau$ gives an order 2 automorphism of N', which we take to be λ. So in terms of Lemma 5.3, we have $N' = H$ and $N = H^{\lambda,-I}$, and the normal subgroup T is taken to be $\langle a \rangle$.

Sylow 2-subgroups:

The first case of a maximal $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, of the form $\langle a \rangle B$, that we will analyze is when $a = 1$, in which case B is a Sylow 2-subgroup of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$. From Lemma 5.1, we know that a Sylow 2-subgroup of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, which we will call $T_2(2n)$, is generated by a Sylow 2-subgroup of $Sp(2n, \mathbb{F}_q)$ fixed by ι, which we denote by $S_2(2n)$, and the element τ. That is,

$$T_2(2n) = \langle S_2(2n), \tau \rangle \cong S_2(2n)^{\iota,-I}.$$

Gow [9, Lemma 3.4] proved the following result on the Frobenius-Schur indicators of the Sylow 2-subgroups of $Sp(2n, \mathbb{F}_q)$.

Lemma 5.4. *Let $S_2(2n)$ by a Sylow 2-subgroup of of $Sp(2n, \mathbb{F}_q)$ and θ any complex irreducible character of $S_2(2n)$. Then $\varepsilon(\theta) = \omega_\theta(-I)$.*

We have the following result for the Frobenius-Schur indicators of the Sylow 2-subgroup of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$.

Lemma 5.5. *Let $S_2(2n)$ be a Sylow 2-subgroup of $Sp(2n, \mathbb{F}_q)$ that is fixed by ι. Let $T_2(2n) \cong S_2(2n)^{\iota,-I}$ be a Sylow 2-subgroup of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$. Then every real-valued irreducible character χ of $T_2(2n)$ satisfies $\varepsilon(\chi) = \omega_\chi(-I)$.*

Proof. From Lemma 5.4, every irreducible character θ of $S_2(2n)$ satisfies $\varepsilon(\chi) = \omega_\chi(-I)$. The result follows by applying Lemma 5.3(ii) to $T_2(2n) \cong S_2(2n)^{\iota,-I}$. □

Subgroups for a real element of $Sp(2n, \mathbb{F}_q)$:

Now we consider a maximal $\mathbb{R}$-elementary subgroup at 2 of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$ of the form $\langle a \rangle B$, where a (not the identity) is an odd order real element of $Sp(2n, \mathbb{F}_q)$, that is, a is conjugate to a^{-1} by an element of $Sp(2n, \mathbb{F}_q)$. Gow [9] considered the maximal $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)$ associated to real elements a. We will use Lemmas 5.1 and 5.2 throughout to make conclusions about the maximal $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$ associated to real elements of $Sp(2n, \mathbb{F}_q)$, based on Gow's analysis of the corresponding subgroups of $Sp(2n, \mathbb{F}_q)$.

So let $a \in Sp(2n, \mathbb{F}_q)$ be a real element of odd order, so that a is conjugate to its inverse in $Sp(2n, \mathbb{F}_q)$. As an element of $GL(2n, \mathbb{F}_q)$, a is of course a real element, and so has the same minimal polynomial as a^{-1}. This self-adjoint polynomial, $m(x)$, may be factored into irreducibles, some of which are self-adjoint, and the other irreducible factors appear with the same multiplicities as their adjoints. This is explained by Wonenburger in [21, Section 1]. So factor $m(x)$ as

$$m(x) = \prod_{i=1}^{e} (p_i(x)\tilde{p}_i(x))^{\mu_i} \prod_{j=1}^{f} (r_j(x))^{\nu_j}, \tag{4}$$

where $\tilde{p}_i(x)$ is the adjoint polynomial of $p_i(x)$, $r_j(x)$ are self-adjoint polynomials, and all of the $p_i(x)$ and $r_j(x)$ are irreducibles of $\mathbb{F}_q[x]$.

Lemma 5.6. *Suppose that $e + f > 1$ in the decomposition of $m(x)$, the minimal polynomial of a, as in Equation (4). Let $M = Sp(2n_1, \mathbb{F}_q) \times \cdots \times Sp(2n_{e+f}, \mathbb{F}_q)$ be the direct product of $e + f$ symplectic groups, where $\sum_{i=1}^{e+f} n_i = n$. Then $N = \langle a \rangle B$ is contained in the subgroup $M^{\iota,-I}$ of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$.*

Proof. Gow [9, Lemma 4.1] proved that the corresponding subgroup of $Sp(2n, \mathbb{F}_q)$, $N' = \langle a \rangle B'$, is contained in the subgroup M of $Sp(2n, \mathbb{F}_q)$, by showing that a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ are contained in M. The lemma follows by applying Lemma 5.2. □

The next case we consider is when $e = 1$ and $f = 0$ in Equation (4). That is, the minimal polynomial of a is the power of the product of an irreducible and its adjoint. In this case we must analyze another subgroup of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, which we construct as follows. Let $\alpha \in \mathbb{F}_{q^2}$ be a square root of -1, and consider the group $\langle GL(n, \mathbb{F}_q), \alpha I \rangle$. We now extend this group by an element, β, whose square is $-I$, and whose conjugation gives the transpose-inverse automorphism. That is, we consider the following group, which contains $GL(n, \mathbb{F}_q)$ as an index 4 normal subgroup:

$$L = \langle GL(n, \mathbb{F}_q), \alpha, \beta \mid \alpha^2 = \beta^2 = -I, \alpha g = g\alpha \text{ for } g \in GL(n, \mathbb{F}_q),$$
$$\beta^{-1} g \beta = {}^t g^{-1} \text{ for } g \in GL(n, \mathbb{F}_q), \beta^{-1}\alpha\beta = \alpha^{-1}\rangle.$$

We have the following for this case.

Lemma 5.7. *Let $e = 1$ and $f = 0$ in the factorization in Equation (4) of $m(x)$, the minimal polynomial of a. Then $N = \langle a \rangle B$ is contained in a subgroup of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ isomorphic to L.*

Proof. The minimal polynomial of a is $m(x) = (p(x)\tilde{p}(x))^{\mu}$, where $p(x)$ is irreducible in $\mathbb{F}_q[x]$ and is not self-adjoint. Consider the subgroup of $Sp(2n, \mathbb{F}_q)$ consisting of elements of the form $\begin{pmatrix} g & \\ & {}^t g^{-1} \end{pmatrix}$, where $g \in GL(n, \mathbb{F}_q)$, which is a subgroup isomorphic to $GL(n, \mathbb{F}_q)$. Gow showed [9, Lemma 4.2] that a can be taken to be in this subgroup, and that the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ is contained in the group generated by this subgroup isomorphic to $GL(n, \mathbb{F}_q)$ and the element $\begin{pmatrix} & I \\ -I & \end{pmatrix}$. If we conjugate a by the element τ, the result is ${}^{\iota}a = a$, since we have taken a to be the block diagonal element above. Then τ, which can be taken to be the element $\begin{pmatrix} -\alpha I & \\ & \alpha I \end{pmatrix}$, is in the centralizer of a in $Sp(2n, \mathbb{F}_q)^{\iota, -I}$, and so in the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)^{\iota, -I}$. This gives all of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)^{\iota, -I}$. So now $N = \langle a \rangle B$ is contained in the subgroup generated by the $GL(n, \mathbb{F}_q)$ subgroup, the element $\beta = \begin{pmatrix} & I \\ -I & \end{pmatrix}$, and the element τ, which is isomorphic to L. □

Lemma 5.8. *Any real-valued irreducible character χ of L satisfies $\varepsilon(\chi) = \omega_{\chi}(-I)$.*

Proof. Let $W = \langle GL(n, \mathbb{F}_q), \alpha \rangle$, and let λ be the transpose-inverse automorphism of $GL(n, \mathbb{F}_q)$, and of W, where ${}^t\alpha^{-1} = \alpha^{-1} = -\alpha$. From Theorem 1.1 and Proposition 2.2, we know that $\varepsilon_{\lambda}(\psi) = 1$ for every irreducible ψ of $GL(n, \mathbb{F}_q)$. Since α is in the center of W, every irreducible character of W is an extension of an irreducible of $GL(n, \mathbb{F}_q)$. If θ is an irreducible of W, which extends the irreducible ψ of $GL(n, \mathbb{F}_q)$, then we may directly calculate that $\varepsilon_{\lambda}(\theta) = \varepsilon_{\lambda}(\psi) = 1$. It also follows from Theorem 1.1, as noted in the comments after Proposition 2.2, that every irreducible character ψ of $GL(n, \mathbb{F}_q)$ satisfies $\varepsilon(\psi) = 1$ or 0. This also holds for any irreducible θ of W, since if θ extends ψ of $GL(n, \mathbb{F}_q)$, either $\varepsilon(\psi) = 0$ and thus $\varepsilon(\theta) = 0$, or $\varepsilon(\psi) = 1$ but $\omega_{\theta}(\alpha)$ is not real and thus $\varepsilon(\theta) = 0$, or $\varepsilon(\psi) = 1$ and $\omega_{\theta}(\alpha) = \pm 1$, in which case $\varepsilon(\theta) = 1$.

We have $L \cong W^{\lambda, -I}$, where λ is the transpose-inverse automorphism. Every irreducible of L is induced or extended from an irreducible of W. An irreducible χ of L is induced from an irreducible of ψ of W if and only if $\varepsilon(\psi) = 0$, by Lemma 2.2, in which case $\varepsilon(\chi) = \omega_\psi(-I)\varepsilon_\lambda(\psi)$, by Lemma 2.3. We have shown above that $\varepsilon_\lambda(\psi) = 1$, and since $\omega_\psi(-I) = \omega_\chi(-I)$, in this case we have $\varepsilon(\chi) = \omega_\chi(-I)$.

If, on the other hand, χ of L is extended from an irreducible ψ of W, then we have $\varepsilon(\psi) = 1$, and $\psi^L = \chi + \chi'$ for another irreducible χ' of L. From Lemma 2.3, and since $\varepsilon(\psi) = \varepsilon_\lambda(\psi) = 1$, we have

$$\varepsilon(\chi) + \varepsilon(\chi') = \varepsilon(\psi) + \omega_\psi(-I)\varepsilon_\lambda(\psi) = 1 + \omega_\psi(-I).$$

Now, $\varepsilon(\chi) = \pm 1$, and at least one of $\varepsilon(\chi)$ or $\varepsilon(\chi')$ must equal 1, since their sum is 2 or 0. Since ψ is the character of a real representation, so is ψ^L, and the representation has a real subrepresentation. This must have a real complement from Maschke's Theorem, and so $\varepsilon(\chi) = \varepsilon(\chi') = 1$. So finally $\omega_\psi(-I) = \omega_\chi(-I) = 1$, and so $\varepsilon(\chi) = \omega_\chi(-I)$. □

The remaining cases in this section deal with the situation $e = 0$ and $f = 1$ in the factorization in Equation (4) of the minimal polynomial of a. So now we assume $m(x) = r(x)^\nu$, where $r(x)$ is an irreducible self-adjoint polynomial. We first deal with the case that $m(x)$ has more than one distinct elementary divisor.

Lemma 5.9. *Suppose $m(x) = r(x)^\nu$ for an irreducible self-adjoint polynomial $r(x)$, and suppose that the distinct elementary divisors of a are*

$$r(x)^{l_1}, \dots, r(x)^{l_t},$$

where $l_1 < \cdots < l_t$, and $t > 1$. Then $N = \langle a \rangle B$ is contained in a subgroup of the form $M^{\iota,-I}$ of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$, where $M = Sp(2n_1, \mathbb{F}_q) \times \cdots \times Sp(2n_t, \mathbb{F}_q)$ is the direct product of smaller symplectic groups.

Proof. Gow [9, Lemma 4.3] showed that the corresponding subgroup $N' = \langle a \rangle B'$ is contained in the subgroup M of $Sp(2n, \mathbb{F}_q)$, by showing a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ are contained in M. The lemma follows from Lemma 5.2. □

We may now assume that a has a single elementary divisor, and we first cover the case that this elementary divisor $r(x)^l$ is such that the self-adjoint irreducible polynomial $r(x)$ is of even degree.

Lemma 5.10. *Let a have the single elementary divisor $r(x)^l$, occurring with multiplicity c, where $r(x)$ is irreducible and self-adjoint of even degree. Let c have 2-adic decomposition*

$$c = 2^{b_1} + \cdots + 2^{b_t},\ b_1 < \cdots < b_t.$$

If $t > 1$, then the subgroup $N = \langle a \rangle B$ of $Sp(2n, \mathbb{F}_q)^{\iota,-I}$ is contained in $M^{\iota,-I}$, where M is the direct product of t smaller symplectic groups.

If $c = 2^b$ and $b \geq 2$, then N is contained in $K^{\iota,-I}$, where K is a wreath product of $Sp(n, \mathbb{F}_q)$ with $\mathbb{Z}/2\mathbb{Z}$.

If $c = 2$, then every irreducible real-valued character χ of N not containing $\langle a \rangle$ in its kernel satisfies $\varepsilon(\chi) = \omega_\chi(-I)$.

If $c = 1$, every irreducible real-valued character χ of N satisfies $\varepsilon(\chi) = \omega_\chi(-I)$.

Proof. Gow [9, Lemma 4.4] proves that if $t > 1$, the corresponding subgroup $N' = \langle a \rangle B'$ of $Sp(2n, \mathbb{F}_q)$ is contained in the subgroup M, and if $c = 2^b$, $b \geq 2$, N' is contained in the subgroup K, by showing that a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ are contained in these subgroups. So by Lemma 5.2, for these cases the subgroups N of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ are contained in $M^{\iota, -I}$ and $K^{\iota, -I}$, respectively.

Gow [9, p. 267-269] shows that for $c = 2$, every irreducible character θ of N' not containing $\langle a \rangle$ in its kernel satisfies $\varepsilon(\theta) = \omega_\theta(-I)$, and that for $c = 1$, N' is isomorphic to a generalized quaternion group with central order 2 element coinciding with $-I$. So for $c = 2$, the conclusion of the lemma in this case follows from applying Lemma 5.3(i). For the case $c = 1$, N' is generalized quaternion, and it is easily checked that all irreducible characters θ of N', except possibly one-dimensional non-real-valued characters, satisfy $\varepsilon(\theta) = \omega_\theta(-I)$. As in the comments after Lemma 5.3, let λ be the automorphism of N' defined by conjugation by $s\tau$, where $N = \langle N', s\tau \rangle$ from Lemma 5.1. All one-dimensional characters satisfy $\varepsilon_\lambda(\theta) \neq -1$, since this indicator can only take the values 0 or 1 for one-dimensionals. Then from Lemma 5.3(ii), every real-valued irreducible χ of N satisfies $\varepsilon(\chi) = \omega_\chi(-I)$. □

The cases that remain deal with the situation that a has a single elementary divisor which is the power of a self-adjoint irreducible $r(x)$ of odd degree. The only possibilities for this are $r(x) = x \pm 1$. But we are assuming that a has odd order, and so it cannot have -1 as an eigenvalue. Therefore we may assume that a is unipotent. The case that is now covered is when a has the single type of elementary divisor of the form $(x-1)^{2l+1}$, which must occur with even multiplicity, since the underlying vector space is of even dimension.

Lemma 5.11. *Suppose a has a single type of elementary divisor $(x-1)^{2l+1}$ occurring with multiplicity $2c$.*

If c is not a power of 2, then N is contained in a subgroup of the form $M^{\iota, -I}$, where M is the direct product of smaller symplectic groups.

If c is a power of 2 greater than 1, then N is contained in a subgroup of the form $K^{\iota, -I}$, where K is the wreath product of $Sp(n, \mathbb{F}_q)$ with $\mathbb{Z}/2\mathbb{Z}$.

If $c = 1$, then N is a subgroup such that every real-valued character χ of N not containing $\langle a \rangle$ in its kernel satisfies $\varepsilon(\chi) = \omega_\chi(-I)$.

Proof. Gow [9, Lemma 4.5] proved that for the first two cases that the corresponding subgroup N' of $Sp(2n, \mathbb{F}_q)$ is contained in a subgroup of the form M and K, respectively, by showing a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ are contained in these subgroups. Lemma 5.2 then applies.

For the case $c = 1$, Gow [9, p. 270] proves that the corresponding N' is such that every irreducible character θ of N' such that $\langle a \rangle$ is not in the kernel of θ satisfies $\varepsilon(\theta) = \omega_\theta(-I)$. The lemma for the case $c = 1$ follows from applying Lemma 5.3. □

The last case for which a is a real element of $Sp(2n, \mathbb{F}_q)$ is the case that a has a single elementary divisor of the form $(x-1)^{2l}$. Since we are assuming that a is a real element of $Sp(2n, \mathbb{F}_q)$, then the elementary divisor $(x-1)^{2l}$ must occur with even multiplicity, which follows from results of Feit and Zuckerman in [5, Lemmas 5.2 and 5.3].

Lemma 5.12. *Suppose a has a single type of elementary divisor $(x-1)^{2l}$ occurring with multiplicity $2c$.*

If $c > 1$ and is not a power of 2, then the subgroup $N = \langle a \rangle B$ of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ is contained in a subgroup of the form $M^{\iota, -I}$, where M is the direct product of smaller symplectic groups.

If $c > 1$ and is a power of 2, N is contained in a subgroup of the form $K^{\iota, -I}$, where K is the wreath product of $Sp(n, \mathbb{F}_q)$ with $\mathbb{Z}/2\mathbb{Z}$.

Proof. Gow [9, Lemma 4.7] proved that N' of $Sp(2n, \mathbb{F}_q)$ is contained in M and K, respectively, by showing that a and the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$ are contained in these subgroups. The lemma follows from Lemma 5.2. □

Lemma 5.13. *Suppose a has a single type of elementary divisor $(x-1)^{2l}$ occurring with multiplicity 2. Then every real-valued character χ of $N = \langle a \rangle B$ not containing $\langle a \rangle$ in its kernel satisfies $\varepsilon(\chi) = \omega_\chi(-I)$.*

Proof. From Gow [9, p.273-274], the element a could lie in one of two conjugacy classes in $Sp(2n, \mathbb{F}_q)$. Let u be a unipotent element of $Sp(n, \mathbb{F}_q)$ that has the elementary divisor $(x-1)^{2l}$ occurring with multiplicity 1. Then a is either conjugate in $Sp(2n, \mathbb{F}_q)$ to an element of the form (u, u^{-1}) or (u, u).

First consider the case that a is of the form (u, u^{-1}). We conveniently change the basis of the underlying vector space so that we may write a as a block diagonal element. Gow calculates that B', the Sylow 2-subgroup of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$, consists of the 8 elements generated by

$$\begin{pmatrix} \pm I & \\ & \mp I \end{pmatrix}, k = \begin{pmatrix} & I \\ I & \end{pmatrix}.$$

Now let s be the element of $Sp(n, \mathbb{F}_q)$ such that $s^{-1}us = {}^{\iota}u^{-1}$, and $(s\tau)^2 = -I$, from Proposition 3.3. The group $N = \langle a \rangle B$ is generated by $N' = \langle a \rangle B'$ and the element $v = \begin{pmatrix} s\tau & \\ & s\tau \end{pmatrix}$. Define the automorphism λ of N' to be conjugation by the element v. Gow showed that every real-valued irreducible character θ of N' not containing $\langle a \rangle$ in its kernel satisfies $\varepsilon(\theta) = \omega_\theta(-I)$. We now show that any other character θ of N' which does not contain $\langle a \rangle$ in its kernel satisfies $\varepsilon_\lambda(\theta) \neq -1$. The group N' contains as an abelian subgroup the following normal subgroup of index 2:

$$A = \left\langle a = \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}, \begin{pmatrix} \pm I & \\ & \mp I \end{pmatrix} \right\rangle.$$

Every irreducible of N' is either induced or extended from an irreducible of A. But since we know that any one-dimensional θ must satisfy $\varepsilon_\lambda(\theta) \neq -1$, we only need to consider irreducibles induced from A. Let ψ be a one-dimensional of A which is nontrivial on $\langle a \rangle$, so that $\psi(a) = \zeta$ for a nontrivial root of unity ζ, and ψ of the order 2 generators of A listed above are ± 1. The automorphism λ sends a to its inverse, and acts trivially on the other generating elements of N' given above. The representation of the character $\theta = \psi^{N'}$ may be given by the matrix representation

$$R(a) = \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix}, \quad R(k) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$

and the elements $(\pm I, \mp I)$ are sent to scalar matrices. It is now clear that $\varepsilon_\lambda(\theta) = 1$. The result for this case now follows from Lemma 5.3(i).

The other case is when a is taken to be the element (u, u), where again we take a basis so that this element can be viewed as block diagonal. Gow calculates [9, p.272-273] that the Sylow 2-subgroup of the centralizer of a in $Sp(2n, \mathbb{F}_q)$ is the Sylow 2-subgroup V of the group generated by elements of the form

$$\begin{pmatrix} \beta I & \gamma I \\ -\gamma I & \beta I \end{pmatrix}, \begin{pmatrix} & I \\ I & \end{pmatrix},$$

where $\beta^2 + \gamma^2 = 1$ in $\mathbb{F}_q$. Then, Gow finds that B', the Sylow 2-subgroup of the $\mathbb{R}$-normalizer of a in $Sp(2n, \mathbb{F}_q)$, is generated by the group V above and the element

$$\begin{pmatrix} bh & ch \\ -ch & bh \end{pmatrix},$$

where $b^2 + c^2 = -1$ in $\mathbb{F}_q$, and where h is the skew-symplectic involution which inverts u, as given by Wonenburger's Theorem 3.1. Gow shows that every real-valued irreducible θ of $N' = \langle a \rangle B'$ which does not contain $\langle a \rangle$ in its kernel satisfies $\varepsilon(\theta) = \omega_\theta(-I)$. Furthermore, he shows that the only characters of N' which do not contain $\langle a \rangle$ in their kernels and that are not real-valued, are induced from certain one-dimensional characters of the group $\langle a \rangle V$, which is an index 2 subgroup of $N' = \langle a \rangle B'$. These two-dimensional characters of N', induced from one-dimensional characters of $\langle a \rangle V$, are the ones we must check satisfy the conditions of Lemma 5.3(i).

Let s be the element of $Sp(n, \mathbb{F}_q)$ such that $s^{-1} u s = {}^{\iota} u^{-1}$, and $(s\tau)^2 = -I$, from Proposition 3.3. Then N is generated by $N' = \langle a \rangle B'$ and the element $(s\tau, s\tau)$. As before, define λ to be the automorphism of N' given by conjugation by the element $(s\tau, s\tau)$. Noting that $s = th$, where $t = \begin{pmatrix} -I & \\ & I \end{pmatrix}$, and h is the skew-symplectic involution inverting u, we may calculate exactly how λ acts on the generators of N'. We may then give a specific matrix representation for each of the characters θ of N' which we must check, and directly see that for each of them, we have $\varepsilon_\lambda(\theta) = 1$, as in the previous case. The lemma then follows by applying Lemma 5.3(i). $\square$

Subgroups for a non-real element of $Sp(2n, \mathbb{F}_q)$:

We now cover the remaining cases of maximal $\mathbb{R}$-elementary subgroups at 2 of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$, and so we let a be an element of $Sp(2n, \mathbb{F}_q)$ of odd order which is not conjugate in $Sp(2n, \mathbb{F}_q)$ to its inverse. It follows from results of Wall [20], as explained by Feit and Zuckerman in [5, Lemmas 5.2 and 5.3], that an element of $Sp(2n, \mathbb{F}_q)$, $q \equiv 3 (\mathrm{mod}\ 4)$, is not real if and only if it has an elementary divisor of the form $(x \pm 1)^{2l}$ which occurs with odd multiplicity. Since we are assuming that a has odd order, then it cannot have -1 as an eigenvalue, and so it has $(x-1)^{2l}$ as an elementary divisor occurring with odd multiplicity. In particular, the minimal polynomial $m(x)$ of a may be factored as $m(x) = f(x)(x-1)^\nu$, where $f(x)$ is self-adjoint and relatively prime to $x - 1$, and $\nu \geq 2$.

Since a is a non-real element of $Sp(2n, \mathbb{F}_q)$, then the maximal $\mathbb{R}$-elementary subgroup at 2 of a in $Sp(2n, \mathbb{F}_q)$ is of the form $N' = \langle a \rangle B'$, where B' is a Sylow 2-subgroup of the centralizer of a in $Sp(2n, \mathbb{F}_q)$. The structure of the centralizers of elements of $Sp(2n, \mathbb{F}_q)$ are described by Wall [20, p.36], and these are the results that we refer to in this section. The results of Huppert [12] give the possible minimal polynomials of an element $g \in Sp(V)$ such that V does not admit a g-invariant orthogonal decomposition. It is from these results that we are able to give the orthogonal decompositions in the proofs of Lemmas 5.15 and 5.16 below. Our first step in this section is to reduce $m(x)$ to a power of $x - 1$.

Lemma 5.14. *Let a have minimal polynomial $m(x) = f(x)(x-1)^\nu$, where $f(x)$ is non-constant. Then $N = \langle a \rangle B$ is contained in a subgroup of the form $(Sp(2n_1, \mathbb{F}_q) \times Sp(2n_2, \mathbb{F}_q))^{\iota, -I}$.*

Proof. As shown by Wonenburger [21, Section 3], the underlying $\mathbb{F}_q$-vector space V on which a acts may be orthogonally decomposed as

$$V \cong \ker(f(x)) \perp \ker((x-1)^\nu).$$

So we may replace a by an element conjugate to a under $Sp(2n, \mathbb{F}_q)$ of the form (a_1, a_2), where the minimal polynomial of a_1 restricted to the $\mathbb{F}_q$-subspace $\ker(f(x))$ of V, is $f(x)$, and $a_1 \in Sp(2n_1, \mathbb{F}_q)$, where $n_1 < n$, and similarly $a_2 \in Sp(2n_2, \mathbb{F}_q)$ with $n_2 < n$, and the minimal polynomial of a_2 restricted to $\ker((x-1)^\nu)$ is $(x-1)^\nu$.

By the results of Wall [20, p. 36], the centralizer of (a_1, a_2) in $Sp(2n, \mathbb{F}_q)$ is the direct product of the centralizer of a_1 in $Sp(2n_1, \mathbb{F}_q)$ and the centralizer of a_2 in $Sp(2n_2, \mathbb{F}_q)$. Since a is a non-real element of $Sp(2n, \mathbb{F}_q)$, then its $\mathbb{R}$-normalizer is exactly its centralizer in $Sp(2n, \mathbb{F}_q)$. Now the result follows from Lemma 5.2. $\square$

We may now assume that the minimal polynomial of a is of the form $m(x) = (x-1)^{\nu}$. The next lemma reduces us further to the case that a has only one elementary divisor.

Lemma 5.15. *Suppose that the distinct elementary divisors of a are*

$$(x-1)^{l_1}, (x-1)^{l_2}, \ldots, (x-1)^{l_t},$$

where $l_1 < l_2 < \cdots < l_t$, $t > 1$. Then $N = \langle a \rangle B$ is contained in a subgroup of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ of the form $(Sp(2n_1, \mathbb{F}_q) \times \cdots \times Sp(2n_t, \mathbb{F}_q))^{\iota, -I}$.

Proof. By applying the main results of Huppert [12], we may orthogonally decompose the underlying $\mathbb{F}_q$-vector space V, non-canonically, as

$$V \cong W_1 \perp \cdots \perp W_t,$$

where a restricted to W_j has the single elementary divisor $(x-1)^{l_j}$. We may then replace a by an element conjugate to it in $Sp(2n, \mathbb{F}_q)$ of the form $(a_1, \ldots, a_t)$, where $a_j \in Sp(2n_j, \mathbb{F}_q)$. From Wall [20, p.36], the centralizer of a in $Sp(2n, \mathbb{F}_q)$ is isomorphic to the product of the centralizers of a_j in $Sp(2n_j, \mathbb{F}_q)$. Therefore we may take a and its centralizer to be in $Sp(2n_1, \mathbb{F}_q) \times \cdots \times Sp(2n_t, \mathbb{F}_q)$, and the result follows from Lemma 5.2. □

We are now reduced to the situation that a has the single elementary divisor $(x-1)^{2l}$ occurring with odd multiplicity, which brings us to the final case to consider.

Lemma 5.16. *Suppose $a \in Sp(2n, \mathbb{F}_q)$ has $(x-1)^{2l}$ as its only elementary divisor, occurring with odd multiplicity c.*

If $c > 1$, then $N = \langle a \rangle B$, is contained in a subgroup of $Sp(2n, \mathbb{F}_q)^{\iota, -I}$ of the form $M^{\iota, -I}$, where M is the direct product of smaller symplectic groups.

If $c = 1$, then $N = \langle a \rangle B$ is such that all of its irreducible characters satisfy $\varepsilon(\chi) = \omega_{\chi}(-I)$.

Proof. According to Wall, the Sylow 2-subgroup of the centralizer of a in $Sp(2n, \mathbb{F}_q)$ is isomorphic to a Sylow 2-subgroup of the orthogonal group $O(c, \mathbb{F}_q)$, with c odd, which we denote by $S(c)$. Because of the structure of $S(c)$, we must consider the cases $c \equiv 1$ and $3 (\mathrm{mod}\ 4)$ separately.

First let $c = 4m+1$. Then by results of Carter and Fong [2, Sec. II], $S(c) \cong S^{+}(4m) \times S(1)$, where $S^{+}(4m)$ is the Sylow 2-subgroup of the split orthogonal group $O^{+}(4m, \mathbb{F}_q)$. By Huppert [12], we may orthogonally decompose the underlying $\mathbb{F}_q$-vector space V on which $a \in Sp(2n, \mathbb{F}_q)$ acts, non-canonically, as

$$V \cong W_{4m} \perp W_1,$$

where a restricted to W_{4m} has the single elementary divisor $(x-1)^{2l}$ with multiplicity $4m$, and a restricted to W_1 has the single elementary divisor $(x-1)^{2l}$ with multiplicity 1. We replace a by an element of the form (a_{4m}, a_1), from this orthogonal decomposition of V, where $a_{4m} \in Sp(8lm, \mathbb{F}_q)$ and

$a_1 \in Sp(2l, \mathbb{F}_q)$. From Gow [9, p. 265], again by applying results of Wall, the centralizers of a_{4m} and a_1 in these smaller symplectic groups have Sylow 2-subgroups isomorphic to $S^+(4m)$ and $S(1)$, respectively. But since the centralizer of a in $Sp(2n, \mathbb{F}_q)$ has Sylow 2-subgroup $S^+(4m) \times S(1)$, now the maximal $\mathbb{R}$-elementary subgroup at 2 in $Sp(2n, \mathbb{F}_q)$ associated with a may be embedded in the subgroup $M \cong Sp(8lm, \mathbb{F}_q) \times Sp(2l, \mathbb{F}_q)$, and so $N = \langle a \rangle B$ is contained in $M^{\iota, -I}$.

Now let $c = 4m + 3$. By Carter and Fong's results [2, Sec. II], $S(c) \cong S^+(4m) \times S(3)$, and $S(3) \cong S^-(2) \times S(1)$, where $S^-(2)$ is the Sylow 2-subgroup of the nonsplit orthogonal group $O^-(2, \mathbb{F}_q)$. We again use the results of Huppert [12] to orthogonally decompose the space V as

$$V \cong W_{4m} \perp W_2 \perp W_1,$$

where a restricted to W_j has the single elementary divisor $(x - 1)^{2l}$ with multiplicity j, and a may be replaced by $(a_{4m}, a_2, a_1) \in Sp(8ml, \mathbb{F}_q) \times Sp(4l, \mathbb{F}_q) \times Sp(2l, \mathbb{F}_q)$. The centralizers of a_j in $Sp(2lj, \mathbb{F}_q)$ have Sylow 2-subgroups, calculated by Gow [9, p. 265, p.272-273] using the results of Wall, which are isomorphic to $S^+(4m)$, $S^-(2)$, and $S(1)$, for $j = 4m, 2$, and 1, respectively. We therefore have $N = \langle a \rangle B \subset M^{\iota, -I}$, where $M = Sp(8ml, \mathbb{F}_q) \times Sp(4l, \mathbb{F}_q) \times Sp(2l, \mathbb{F}_q)$.

Finally, we consider the case $c = 1$. By Wall [20, p.36], the Sylow 2-subgroup of the centralizer of a in $Sp(2n, \mathbb{F}_q)$ consists of only 2 elements, which are $\pm I$. If s is the element of $Sp(2n, \mathbb{F}_q)$ such that $s^{-1} a s = {}^{\iota}a^{-1}$, from Proposition 3.3, then we have $B = \langle \pm I, s\tau \rangle$. So N contains as an index 2 subgroup the cyclic group $C \cong \langle a \rangle \times \langle \pm I \rangle$, on which the element $s\tau$ acts by inversion, and $(s\tau)^2 = -I$. We therefore have $N \cong C^{\lambda, -I}$, where λ is the automorphism which inverts the elements of C. Every irreducible χ of N is induced or extended from an irreducible θ of C, and θ^N is irreducible if and only if ${}^{\lambda}\theta \neq \theta$, from Lemma 2.2. But ${}^{\lambda}\theta = \overline{\theta}$, and so θ^N is irreducible if and only if θ is not real-valued. We also see that since θ is one-dimensional that $\varepsilon_{\lambda}(\theta) = 1$, and $\varepsilon(\theta) = 1$ when θ is real-valued, and $\varepsilon(\theta) = 0$ otherwise. So when $\varepsilon(\theta) = 0$, if $\chi = \theta^N$, we have by Lemma 2.3 and the comments above,

$$\varepsilon(\chi) = \varepsilon(\theta) + \omega_{\theta}(-I)\varepsilon_{\lambda}(\theta) = 0 + \omega_{\theta}(-I) = \omega_{\chi}(-I).$$

When $\varepsilon(\theta) = 1$, $\theta^N = \chi + \chi'$, where χ and χ' are extensions of θ. Again by Lemma 2.3 and the comments above, we have

$$\varepsilon(\chi) + \varepsilon(\chi') = \varepsilon(\theta) + \omega_{\theta}(-I)\varepsilon_{\lambda}(\theta) = 1 + \omega_{\theta}(-I).$$

At least one of $\varepsilon(\chi)$ and $\varepsilon(\chi')$ is 1, but then the other must be 1, since θ^N is a real representation, and real subrepresentations will have a real complement. This forces $\omega_{\theta}(-I) = \omega_{\chi}(-I) = 1$. We therefore have any irreducible χ of N satisfies $\varepsilon(\chi) = \omega_{\chi}(-I)$. □

6. Main Results

Theorem 6.1. *Let $G = Sp(2n, \mathbb{F}_q)$, and let $\iota : G \to G$ be the order 2 automorphism of G defined by*

$$^{\iota}g = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix} g \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}.$$

Let $G^{\iota,-I} = \langle G, \tau \mid \tau^2 = -I, \tau^{-1} g \tau = \, ^{\iota}g \text{ for all } g \in G \rangle$.

(i) If $q \equiv 3(mod\ 4)$, then every complex irreducible representation ϕ of $G^{\iota,-I}$, with central character ω_ϕ, satisfies $\varepsilon(\phi) = \omega_\phi(-I)$

(ii) If q is odd, then every complex irreducible representation π of G satisfies $\varepsilon_\iota(\pi) = 1$.

Proof. (i): The proof is by induction on n. The case for $n = 1$ is proven in Section 4, and we have the induction hypothesis that the result is true for any $m < n$. Let ϕ be any complex irreducible representation of $G^{\iota,-I}$ with character χ. Then χ must be real-valued by Proposition 3.5. By Proposition 4.1(i), there is an irreducible real-valued character ψ of an $\mathbb{R}$-elementary subgroup at 2, N of $G^{\iota,-I}$, such that $\langle \chi|_N, \psi \rangle$ is odd and $\varepsilon(\chi) = \varepsilon(\psi)$. By Proposition 4.1(ii) and (iv), we may choose N to either be a Sylow 2-subgroup of $G^{\iota,-I}$, or N to be a maximal $\mathbb{R}$-elementary subgroup at 2, $N = \langle a \rangle B$, with a a real element of odd order, and with ψ satisfying $\langle a \rangle \not\subset \ker(\psi)$.

In Section 5, we consider every possible maximal $\mathbb{R}$-elementary subgroup at 2 with a of odd order. If N is a Sylow 2-subgroup, Lemma 5.5 says that every real-valued irreducible character ψ of N satisfies $\varepsilon(\psi) = \omega_\psi(-I)$. When N is not a Sylow 2-subgroup, in some cases, we prove that every irreducible real-valued character ψ of N satisfying $\langle a \rangle \not\subset \ker(\psi)$ (we don't even need this assumption in some cases) is such that $\varepsilon(\psi) = \omega_\psi(-I)$. So if N is one of these subgroups, and if ψ is the character of N such that $\langle \chi|_N, \psi \rangle$ is odd, then $\varepsilon(\chi) = \varepsilon(\psi)$, and $\omega_\psi(-I) = \omega_\chi(-I)$ since ψ is a constituent of $\chi|_N$. So in these cases, we have $\varepsilon(\chi) = \omega_\chi(-I)$.

In other cases, we prove that an isomorphic copy of N is contained in a subgroup of the form $M^{\iota,-I}$ or $K^{\iota,-I}$, where M is a product of symplectic groups of smaller dimension, and K is a wreath product of a symplectic group of half the dimension with $\mathbb{Z}/2\mathbb{Z}$. In the case of Lemma 5.7, we prove an isomorphic copy of N is contained in a subgroup L, which contains $GL(n, \mathbb{F}_q)$ as a subgroup of index 4. By Proposition 4.1(iii), if N is contained in a subgroup of $G^{\iota,-I}$, then there is a real-valued irreducible character θ of that subgroup such that $\langle \theta^{G^{\iota,-I}}, \chi \rangle$ is odd and $\varepsilon(\chi) = \varepsilon(\psi) = \varepsilon(\theta)$. But now, by Propositions 4.2 and 4.4 along with the induction hypothesis, and by Lemma 5.8, every real-valued irreducible θ of a group of the form $M^{\iota,-I}$, $K^{\iota,-I}$, or L, satisfies $\varepsilon(\theta) = \omega_\theta(-I)$. So if N falls into this category, $\varepsilon(\chi) = \varepsilon(\theta) = \omega_\theta(-I) = \omega_\chi(-I)$. This exhausts all possible cases for N, and the result is obtained.

(ii): First assume $q \equiv 1 (\mathrm{mod}\ 4)$. Then by Gow's Theorem 1.2, we have $\varepsilon(\pi) = \omega_\pi(-I)$ for every irreducible π of $Sp(2n, \mathbb{F}_q)$. The automorphism ι is inner by the element $\begin{pmatrix} -\alpha I_n & \\ & \alpha I_n \end{pmatrix}$, where $\alpha^2 = -1$, and this matrix has square $-I$. So by Lemma 2.1, $\varepsilon_\iota(\pi) = \omega_\pi(-I)\varepsilon(\pi) = \omega_\pi(-I)^2 = 1$. If $q \equiv 3 (\mathrm{mod}\ 4)$, then first we have $\varepsilon_\iota(\pi) = \pm 1$ by Proposition 3.3. Now by part (i) and Proposition 2.2, we have $\varepsilon_\iota(\pi) = 1$ for every irreducible π of G. $\square$

Theorem 6.2. *Let $G = GSp(2n, \mathbb{F}_q)$, where q is odd, and μ is the similitude character. Let $\iota : G \to G$ be the inner order 2 automorphism of G defined by conjugation by the skew-symplectic element as in Theorem 6.1. Let $\kappa : G \to G$ be the order 2 automorphism of G defined by*

$$^{\kappa}g = \mu(g)^{-1}g.$$

Let $\sigma : G \to G$ be the order 2 automorphism that is the composition of ι with κ, $\sigma = \iota \circ \kappa$. Then every complex irreducible representation π of G satisfies $\varepsilon_\kappa(\pi) = \omega_\pi(-I)$, and equivalently satisfies $\varepsilon_\sigma(\pi) = 1$.

Proof. First, by Proposition 3.4, we have $\varepsilon_\kappa(\pi) = \pm 1$ for every irreducible (π, V) of G. Now let $t = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}$, so $^{\kappa}t = -t$, and let χ be the character of π. Then

$$\varepsilon_\sigma(\pi) = \frac{1}{|G|}\sum_{g \in G} \chi(g\, {}^{\sigma}g) = \frac{1}{|G|}\sum_{g \in G} \chi(-gt\, {}^{\kappa}(gt)) = \omega_\pi(-I)\varepsilon_\kappa(\pi).$$

Since $\varepsilon_\sigma(\pi) = \pm 1$, we have a nondegenerate bilinear form $B_\sigma : V \times V \to \mathbb{C}$, unique up to scalar, such that for all $g \in G$, $u, v \in V$,

$$B_\sigma(\pi(g)u, {}^{\sigma}\pi(g)v) = B_\sigma(u, v) \quad \text{and} \quad B_\sigma(u, v) = \varepsilon_\sigma(\pi)B_\sigma(v, u).$$

Let Z be the center of $G = GSp(2n, \mathbb{F}_q)$ consisting of scalar matrices, and let $H = Z \cdot Sp(2n, \mathbb{F}_q)$. Then H is an index 2 subgroup of G consisting of every element whose similitude factor is a square in $\mathbb{F}_q^\times$. Every irreducible representation ϕ of $Sp(2n, \mathbb{F}_q)$ may be extended to an irreducible representation of H by just extending the central character to Z, and so any irreducible representation of H restricted to $Sp(2n, \mathbb{F}_q)$ is irreducible. Since H is an index 2 subgroup of G, every irreducible representation π of G restricted to H is either irreducible or the direct sum of 2 distinct irreducibles.

First assume that (π, V) of G restricts to an irreducible (π', V) of H. Then π' restricted to $Sp(2n, \mathbb{F}_q)$ is some irreducible ϕ. Note that for $g \in Sp(2n, \mathbb{F}_q)$, we have ${}^{\sigma}g = {}^{\iota}g$. Then for any $g \in Sp(2n, \mathbb{F}_q)$ and $u, v \in V$, we have

$$B_\sigma(\pi(g)u, {}^{\sigma}\pi(g)v) = B_\sigma(\phi(g)u, {}^{\iota}\phi(g)v) = B_\sigma(u, v).$$

From Theorem 6.1(ii), we know that $\varepsilon_\iota(\phi) = 1$, so there is a nondegenerate symmetric bilinear form, unique up to scalar, satisfying

$$B_\iota(\phi(g)u, {}^{\iota}\phi(g)v) = B_\iota(u, v),$$

for all $g \in Sp(2n, \mathbb{F}_q)$, $u, v \in V$. So then B_σ must be a scalar multiple of B_ι, and therefore must also be symmetric. Then we have $\varepsilon_\sigma(\pi) = 1$.

Now assume that the irreducible (π, V) of G, when restricted to H, is isomorphic to the direct sum of two irreducible representations (π_1, V_1) and (π_2, V_2), which restrict to $Sp(2n, \mathbb{F}_q)$ to give the irreducibles (ϕ_1, V_1) and (ϕ_2, V_2), respectively. Now for any $g \in Sp(2n, \mathbb{F}_q)$, and $u, v \in V_1$, we have

$$B_\sigma(\phi_1(g)u, {}^\iota\phi_1(g)v) = B_\sigma(u, v).$$

Again from Theorem 6.1(ii), $\varepsilon_\iota(\phi_1) = 1$, and so there is a symmetric nondegenerate bilinear form B_ι on V_1, unique up to scalar, with the property of ι-twisted-invariance under $Sp(2n, \mathbb{F}_q)$. Then if B_σ restricted to $V_1 \times V_1$ is nondegenerate, it would have to be a scalar multiple of B_ι, and so B_σ would be symmetric on $V_1 \times V_1$. But since B_σ is either symmetric or skew-symmetric on all of $V \times V$, then being nondegenerate and symmetric on a subspace forces it to be symmetric everywhere. So now we must show B_σ is nondegenerate on $V_1 \times V_1$.

For $g \in Sp(2n, \mathbb{F}_q)$, $u \in V_1$, and $v \in V_2$, we have

$$B_\sigma(\pi(g)u, {}^\sigma\pi(g)v) = B_\sigma(\phi_1(g)u, {}^\iota\phi_2(g)v) = B_\sigma(u, v).$$

So if B_σ is nondegenerate on $V_1 \times V_2$, then we would have $\hat{\phi}_1 \cong {}^\iota\phi_2$. But ${}^\iota\phi_2 \cong \hat{\phi}_2$, and so we would have $\phi_2 \cong \phi_1$. This would imply that $\pi_1 \cong \pi_2$, since the central characters of π_1 and π_2 agree with the central character of π. But we cannot have π restricted to an index 2 subgroup be the direct sum of 2 isomorphic representations, by [14, Corollary 6.19]. So now B_σ must be zero on $V_1 \times V_2$, by Schur's Lemma, which means B_σ must be nondegenerate on $V_1 \times V_1$, since B_σ is nondegenerate on $V \times V$ and $V = V_1 \oplus V_2$. Therefore B_σ is symmetric, and $\varepsilon_\sigma(\pi) = 1$.

So for every irreducible π of $GSp(2n, \mathbb{F}_q)$, we have $\varepsilon_\sigma(\pi) = 1$, and since $\varepsilon_\sigma(\pi) = \omega_\pi(-I)\varepsilon_\kappa(\pi)$, we also have $\varepsilon_\kappa(\pi) = \omega_\pi(-I)$. □

Corollary 6.1. *Let q be odd. The sum of the degrees of the complex irreducible characters of $Sp(2n, \mathbb{F}_q)$ is equal to*

$$\frac{|Sp(2n, \mathbb{F}_q)|}{|GL(n, \mathbb{F}_q)|} = q^{n(n+1)/2}(q^n + 1) \cdots (q + 1)$$

$$= \text{the number of skew-symplectic symmetric matrices in } GSp(2n, \mathbb{F}_q).$$

The sum of the degrees of the complex irreducible characters of $GSp(2n, \mathbb{F}_q)$ is equal to

$$\frac{|GSp(2n, \mathbb{F}_q)|}{2|GL(n, \mathbb{F}_q)|} + \frac{|GSp(2n, \mathbb{F}_q)|}{2|U(n, \mathbb{F}_{q^2})|}$$

$$= \text{the number of symmetric matrices in } GSp(2n, \mathbb{F}_q).$$

Proof. For $G = Sp(2n, \mathbb{F}_q)$, we have from Theorem 6.1(ii) and Proposition 2.1(ii), with ι defined as before,

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = |\{g \in Sp(2n, \mathbb{F}_q) \mid g\, {}^\iota g = I\}|.$$

We have ${}^{\iota}g = tgt$, where $t \in GSp(2n, \mathbb{F}_q)$ and $\mu(t) = -1$, so that $g\, {}^{\iota}g = (gt)^2$, where $\mu(gt) = -1$. So now

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = |\{h \in GSp(2n, \mathbb{F}_q) \mid h^2 = I \text{ and } \mu(h) = -1\}|.$$

From Proposition 3.2, the elements of $GSp(2n, \mathbb{F}_q)$ satisfying $h^2 = I$ and $\mu(h) = -1$ form a single conjugacy class whose centralizer contains an index $q-1$ subgroup isomorphic to $GL(n, \mathbb{F}_q)$. So the size of the conjugacy class is

$$\frac{|GSp(2n, \mathbb{F}_q)|}{(q-1)|GL(n, \mathbb{F}_q)|} = \frac{|Sp(2n, \mathbb{F}_q)|}{|GL(n, \mathbb{F}_q)|} = q^{n(n+1)/2}(q^n + 1) \cdots (q+1).$$

For $G = GSp(2n, \mathbb{F}_q)$, we have from Theorem 6.2 and Proposition 2.1(ii) that

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = |\{g \in GSp(2n, \mathbb{F}_q) \mid g\, {}^{\sigma}g = I\}|,$$

where $g\, {}^{\sigma}g = \mu(g)^{-1}(gt)^2$, where $-\mu(gt) = \mu(g)$. So now

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = |\{h \in GSp(2n, \mathbb{F}_q) \mid h^2 = -\mu(h)I\}|.$$

From Proposition 3.2, for each choice of $\mu(h)$, there is a single conjugacy class of elements satisfying $h^2 = -\mu(h)I$, with centralizer having a subgroup of index $q-1$ isomorphic to $GL(n, \mathbb{F}_q)$ if $-\mu(h)$ is a square, and $U(n, \mathbb{F}_{q^2})$ if $-\mu(h)$ is not a square. Since half of the elements of $\mathbb{F}_q^{\times}$ are squares and half nonsquares, we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = \frac{|GSp(2n, \mathbb{F}_q)|}{2|GL(n, \mathbb{F}_q)|} + \frac{|GSp(2n, \mathbb{F}_q)|}{2|U(n, \mathbb{F}_{q^2})|}.$$

Finally, if $g \in GSp(2n, \mathbb{F}_q)$, and $J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$, then Jg is symmetric if and only if $Jg = -{}^{t}gJ$, since ${}^{t}J = -J$. Multiplying by g on the right, and using ${}^{t}gJg = \mu(g)J$, gives $g^2 = -\mu(g)I$. So now the sum of the degrees for $GSp(2n, \mathbb{F}_q)$ is the number of symmetric matrices in $GSp(2n, \mathbb{F}_q)$, and the sum of the degrees for $Sp(2n, \mathbb{F}_q)$ is the number of skew-symplectic symmetric matrices in $GSp(2n, \mathbb{F}_q)$. □

Remark. The methods in this paper do not work for characteristic 2. However, Gow proved [7] that if q is a power of 2, then every character of $Sp(2n, \mathbb{F}_q)$ is real-valued. From the remark of Prasad at the end of [18, Lemma 1], one may conclude that if q is even, every generic representation of $Sp(2n, \mathbb{F}_q)$ is real. For $SL(2, \mathbb{F}_q)$ and q even, one may check that all of the representations are real, and the sum of the degrees is equal to the number of symmetric matrices (and involutions) in the group.

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