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A note on orthogonal similitude groups

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Let V be a vector space over the field F such that $\mathrm{char}(F) \neq 2$, and let V have a symmetric nondegenerate bilinear form. Let $\mathrm{GO}(V)$ be the orthogonal similitude group for this symmetric form, with similitude character μ. We prove that if $g \in \mathrm{GO}(V)$ with $\mu(g) = \beta$, then $g = t_1 t_2$ where t_1 is an orthogonal involution, and t_2 is such that $t_2^2 = \beta I$ and $\mu(t_2) = \beta$. As an application, we obtain an expression for the sum of the degrees of the irreducible characters of $\mathrm{GO}(n, \mathbb{F}_q)$ for odd q.

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1. Introduction

This note is an addendum to [1], where we obtain a factorization in the symplectic similitude group. In Theorem 1 below, we obtain a factorization in the group of orthogonal similitudes $\mathrm{GO}(V)$, where V is an F-vector space with $\mathrm{char}(F) \neq 2$, and the similitude character is μ. The method is the same as in [1], and the notation established there is freely used. In the proof of Theorem 1, we refer to [1] to all parts that immediately apply to the orthogonal case, while any changes that are needed in the proof are given specifically.

As an application of Theorem 1, we use a result of Gow [2] on the orthogonal group over a finite field to obtain information on the characters of the orthogonal similitude group over a finite field, as given in Theorem 2 and Corollary 2. In a paper to appear by Adler and Prasad [3], Corollary 1 is used to prove a theorem on p-adic groups. In particular, if V is a vector space over a p-adic field, Adler and Prasad prove that any irreducible admissible representation of $\mathrm{GO}(V)$ restricted to $\mathrm{O}(V)$ is multiplicity free, and they also prove the corresponding statement for the symplectic similitude group.

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2. The main theorem

Let V be an F-vector space, with $\mathrm{char}(F) \neq 2$, equipped with a nondegenerate bilinear symmetric form $\langle \cdot, \cdot \rangle : V \times V \to F$. Then the *orthogonal group of similitudes* of V with respect to this form is the group $\mathrm{GO}(V) = \{g \in \mathrm{GL}(V) \colon \langle gv, gw \rangle = \mu(g)\langle v, w \rangle$ for some $\mu(g) \in F^\times$ for all $v, w \in V\}$. Then $\mu : \mathrm{GO}(V) \to F^\times$ is a multiplicative character called the *similitude character*, and the *orthogonal group* is $\mathrm{O}(V) = \ker(\mu)$.

Theorem 1 *Let g be an element of* $\mathrm{GO}(V)$ *satisfying $\mu(g) = \beta$. Then we may factor g as $g = t_1 t_2$, where t_1 is an orthogonal involution and t_2 satisfies $t_2^2 = \beta I$ and $\mu(t_2) = \beta$.*

Proof Wonenburger [4] proved that any element of $\mathrm{O}(V)$ is the product of two orthogonal involutions. So if $\mu(g) = \beta$ is a square in F, then the theorem follows directly from Wonenburger's result. So we assume β is not a square.

As in [1], for any monic polynomial $f \in F[x]$ of degree d, define the *β-adjoint* of f to be

$$\hat{f}(x) = f(0)^{-1} x^d f(\beta/x),$$

and define a monic polynomial to be *self-β-adjoint* if $\hat{f} = f$. Then, for any $g \in \mathrm{GO}(V)$, the minimal polynomial of g is self-β-adjoint. All of the results in sections 2 and 3 of [1] are valid for transformations g which are self-β-adjoint, as the proofs only use this fact. These results reduce us to looking at the case that either g is a cyclic transformation for V, that is V is generated by vectors of the form $g^i v$ for some $v \in V$, or the case that g has minimal polynomial of the form $q(x)^s$, where $q(x)$ is an irreducible self-β-adjoint polynomial.

We deal with the cyclic case first. In [1, Proposition 3(i)], we prove that if $g \in \mathrm{GL}(V)$ is a cyclic transformation with self-β-adjoint minimal polynomial, ignoring any inner product structure, then we can factor $g = t_1 t_2$ such that $t_1^2 = I$ and $t_2^2 = \beta I$. This is proven as follows. If V is cyclic for the vector v, then we let P be the space spanned by vectors of the form $(g^i + \beta^i g^i)v$ and let Q be the space spanned by the vectors of the form $(g^i - \beta^i g^i)v$. Then $V = P \oplus Q$, and the transformation having P as its $+1$ eigenspace and Q as its -1 eigenspace is exactly the involution t_1 that we seek. For the case that $g \in \mathrm{GO}(V)$, we must show that this t_1 is orthogonal. Let $(g^i + \beta^i g^i)v \in P$ and $(g^j - \beta^j g^j)v \in Q$. Then we have:

$$\begin{aligned}
&\left\langle (g^i + \beta^i g^{-i})v,\ (g^j - \beta^j g^{-j})v \right\rangle \\
&= \left\langle (g^i + \beta^i g^{-i})v,\ g^j v \right\rangle - \left\langle g^i v,\ \beta^j g^{-j} v \right\rangle - \left\langle \beta^i g^{-i} v,\ \beta^j g^{-j} v \right\rangle \\
&= \left\langle (g^i + \beta^i g^{-i})v,\ g^j v \right\rangle - \left\langle g^j v,\ \beta^i g^{-i} v \right\rangle - \left\langle g^j v,\ g^i v \right\rangle \\
&= \left\langle (g^i + \beta^i g^{-i})v,\ g^j v \right\rangle - \left\langle g^j v,\ (g^i + \beta^i g^{-i})v \right\rangle \\
&= 0.
\end{aligned}$$

So P and Q are mutually orthogonal. Now let u and u' be any two vectors in $V = P \oplus Q$. Write $u = w + y$, $u' = w' + y'$, where $w, w' \in P$ and $y, y' \in Q$. We compute $\langle t_1 u, t_1 u' \rangle$:

$$\begin{aligned}
\left\langle t_1 u, t_1 u' \right\rangle &= \left\langle t_1(w + y),\ t_1(w' + y') \right\rangle = \left\langle w - y,\ w' - y' \right\rangle \\
&= \left\langle w, w' \right\rangle + \left\langle y, y' \right\rangle - \left\langle y, w' \right\rangle - \left\langle w, y' \right\rangle \\
&= \left\langle w, w' \right\rangle + \left\langle y, y' \right\rangle.
\end{aligned}$$

While computing $\langle u, u' \rangle$ gives us:

$$\begin{aligned}\langle u, u' \rangle &= \langle w + y,\ w' + y' \rangle \\ &= \langle w, w' \rangle + \langle y, y' \rangle + \langle y, w' \rangle + \langle w, y' \rangle \\ &= \langle w, w' \rangle + \langle y, y' \rangle.\end{aligned}$$

Therefore, we have $\langle t_1 u,\ t_1 u' \rangle = \langle u, u' \rangle$, and t_1 is orthogonal. Since g satisfies $\mu(g) = \beta$, then $t_2 = t_1 g$ satisfies $\mu(t_2) = \beta$.

We now deal with the case that the minimal polynomial of g is of the form $q(x)^s$, where $q(x)$ is irreducible and self-β-adjoint. It follows from [1, Lemmas 4 and 5] and the cyclic case above that we may assume that V is the sum of two cyclic spaces, and further we may assume that for any $u_1 \in V$ satisfying $q(g)^{s-1} u_1 \neq 0$, the cyclic space U_1 generated by u_1 is degenerate, and for an appropriate $u_2 \in V$, we have $V = U_1 \oplus U_2$ where U_2 is the cyclic space generated by u_2. We follow the proof of [1, Proposition 3(iii)]. We may write $U_1 = P_1 \oplus Q_1$ where P_1 is spanned by vectors of the form $(g^k + \beta^k g^{-k}) u_1$ and Q_1 is spanned by vectors of the form $(g^k - \beta^k g^{-k}) u_1$.

There are two different cases, the first is when either $q(x)$ is relatively prime to $x^2 - \beta$ or $q(x) = x^2 - \beta$ and s is odd. In this case, we find a $u_2 \in Q_1^{\perp}$, and $V = U_1 \oplus U_2$ where U_2 is cyclically generated by u_2. Then $U_2 = P_2 \oplus Q_2$, where P_2 is spanned by vectors of the form $(g^k + \beta^k g^{-k}) u_2$ and Q_2 is spanned by vectors of the form $(g^k - \beta^k g^{-k}) u_2$. Letting $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$, we are able to show that if t_1 is the involution with P as its $+1$ eigenspace and Q as its -1 eigenspace, then $(t_1 g)^2 = \beta I$ (for the symplectic group, we actually show this in the second case of Proposition 3(iii)). We need to show that t_1 is orthogonal. From the cyclic case above, we have $P_i \perp Q_i$ for $i = 1, 2$. In the proof of [1, Proposition 3(iii)], we show that $P_i \perp Q_j$ for $i \neq j$. So now $P \perp Q$, and from the argument in the cyclic case above, we have that t_1 is orthogonal, and so $t_2 = t_1 g$ satisfies $\mu(t_2) = \beta$.

In the case that $q(x) = x^2 - \beta$ and s is even, we are able to find a $u_2 \in P_1^{\perp}$ such that $V = U_1 \oplus U_2$, where U_2 is the space cyclically generated by u_2. We define P_2 and Q_2 as before. We let $P = P_1 \oplus Q_2$ be the $+1$ eigenspace and $Q = P_2 \oplus Q_1$ be the -1 eigenspace of an involution t_1, and this satisfies $(t_1 g)^2 = \beta I$. To show t_1 is orthogonal, we need only show that $P \perp Q$ and appeal to the cyclic case above. We have already shown $P_i \perp Q_i$ for $i = 1, 2$, so now we need $P_1 \perp P_2$ and $Q_1 \perp Q_2$. We have:

$$\begin{aligned}&\langle (g^k \pm \beta^k g^{-k}) u_1,\ (g^l \pm \beta^l g^{-l}) u_2 \rangle \\ &\quad = \langle (g^k \pm \beta^k g^{-k}) u_1,\ g^l u_2 \rangle \pm \langle (g^k \pm \beta^k g^{-k}) u_1,\ \beta^l g^{-l} u_2 \rangle \\ &\quad = \langle \beta^l g^{-l} (g^k \pm \beta^k g^{-k}) u_1,\ u_2 \rangle \pm \langle g^l (g^k \pm \beta^k g^{-k}) u_1,\ u_2 \rangle \\ &\quad = \pm \langle (g^l \pm \beta^l g^{-l})(g^k \pm \beta^k g^{-k}) u_1,\ u_2 \rangle \\ &\quad = 0,\end{aligned}$$

since

$$\begin{aligned}&(g^l \pm \beta^l g^{-l})(g^k \pm \beta^k g^{-k}) u_1 \\ &\quad = (g^{l+k} + \beta^{l+k} g^{-(l+k)}) u_1 \pm \beta^k (g^{l-k} + \beta^{l-k} g^{-(l-k)}) u_1 \in P_1\end{aligned}$$

and $u_2 \in P_1^{\perp}$. So now as before, we have t_1 orthogonal. This exhausts all cases, and the theorem is proved. ■

COROLLARY 1 *Any element of $g \in \mathrm{GO}(V)$ is conjugate to $\mu(g)g^{-1}$ by an orthogonal involution.*

3. Application over a finite field

Let G be a finite group with an order 2 automorphism ι, let (π, V) be an irreducible complex representation, and let $\hat{\pi}$ denote the contragredient representation. If ${}^{\iota}\pi \cong \hat{\pi}$, where ${}^{\iota}\pi(g) = \pi({}^{\iota}g)$, then we obtain a bilinear form $B_{\iota} : V \times V \to \mathbb{C}$ satisfying

$$B_{\iota}(\pi(g)v, {}^{\iota}\pi(g)w) = B_{\iota}(v, w) \quad \text{for every } v, w \in V. \qquad (*)$$

By Schur's Lemma, this bilinear form is unique up to scalar, which means we have, for all $v, w \in V$,

$$B_{\iota}(v, w) = \varepsilon_{\iota}(\pi) B_{\iota}(w, v),$$

where $\varepsilon_{\iota}(\pi) = \pm 1$. That is, B_{ι} is either symmetric or skew-symmetric. Since the character of $\hat{\pi}$ is $\bar{\chi}$ if χ is the character of π, then ${}^{\iota}\pi \cong \hat{\pi}$ is equivalent to ${}^{\iota}\chi = \bar{\chi}$.

Let $\mathbb{F}_q$ be the finite field of q elements, and let q be odd. We let $\mathrm{O}(n, \mathbb{F}_q)$ be the orthogonal group for any symmetric form (split or nonsplit) for an $\mathbb{F}_q$-vector space. Let $\mathrm{GO}(n, \mathbb{F}_q)$ be the corresponding orthogonal similitude group with similitude character μ.

PROPOSITION 1 *Let q be odd and $G = \mathrm{GO}(n, \mathbb{F}_q)$. Define ι to be the order 2 automorphism of G that acts as ${}^{\iota}g = \mu(g)^{-1}g$. Then every irreducible representation π of G satisfies ${}^{\iota}\pi \cong \hat{\pi}$, that is, $\varepsilon_{\iota}(\pi) = \pm 1$.* ■

Proof From Corollary 2, we have g is conjugate to $\mu(g)g^{-1}$, and so g^{-1} is always conjugate to ${}^{\iota}g$. Thus every character satisfies ${}^{\iota}\chi = \bar{\chi}$, and so for every π we have $\varepsilon_{\iota}(\pi) = \pm 1$. ■

Gow [2] showed that for q odd, every irreducible representation of $\mathrm{O}(n, \mathbb{F}_q)$ is self-dual and orthogonal. This corresponds to ι being the identity automorphism, and $\varepsilon_{\iota}(\pi) = \varepsilon(\pi) = 1$. We are able to apply his result in order to obtain the following stronger version of Proposition 1.

THEOREM 2 *Let q be odd and $G = \mathrm{GO}(n, \mathbb{F}_q)$. Define ι to be the order 2 automorphism of G that acts as ${}^{\iota}g = \mu(g)^{-1}g$. Then every irreducible representation π of G satisfies $\varepsilon_{\iota}(\pi) = 1$.*

Proof Since $\varepsilon_{\iota}(\pi) = \pm 1$ from Proposition 1, then we have a bilinear form B_{ι} as in (*).

Let Z be the center of $G = \mathrm{GO}(n, \mathbb{F}_q)$ consisting of scalar matrices, and let $H = Z \cdot \mathrm{O}(n, \mathbb{F}_q)$. Then H is an index 2 subgroup of G consisting of elements whose similitude factor is a square in $\mathbb{F}_q^{\times}$. Every irreducible representation ϕ of $\mathrm{O}(n, \mathbb{F}_q)$ may be extended to an irreducible representation of H by just extending the central character to Z, and so any irreducible representation of H restricted to $\mathrm{O}(n, \mathbb{F}_q)$

is irreducible. Since H is an index 2 subgroup of G, every irreducible representation π of G restricted to H is either irreducible or the direct sum of 2 distinct irreducibles.

First assume that (π, V) of G restricts to an irreducible (π', V) of H. Then π' restricted to $\mathrm{O}(n, \mathbb{F}_q)$ is some irreducible ϕ. Note that for $g \in \mathrm{O}(n, \mathbb{F}_q)$, we have ${}^{\iota}g = g$. Then for any $g \in \mathrm{O}(n, \mathbb{F}_q)$ and $u, v \in V$, we have

$$B_\iota(\pi(g)u, {}^{\iota}\pi(g)v) = B_\iota(\phi(g)u, \phi(g)v) = B_\iota(u, v).$$

From Gow's result, we know that $\varepsilon(\phi) = 1$, so there is a nondegenerate symmetric bilinear form, unique up to scalar, satisfying

$$B(\phi(g)u, \phi(g)v) = B(u, v),$$

for all $g \in \mathrm{O}(n, \mathbb{F}_q)$, $u, v \in V$. So then B_ι must be a scalar multiple of B, and therefore must also be symmetric. Then we have $\varepsilon_\iota(\pi) = 1$.

Now assume that the irreducible (π, V) of G, when restricted to H, is isomorphic to the direct sum of two irreducible representations (π_1, V_1) and (π_2, V_2), which restrict to $\mathrm{O}(n, \mathbb{F}_q)$ to give the irreducibles (ϕ_1, V_1) and (ϕ_2, V_2), respectively. Now for any $g \in \mathrm{O}(n, \mathbb{F}_q)$, and $u, v \in V_1$, we have

$$B_\iota(\phi_1(g)u, \phi_1(g)v) = B_\iota(u, v).$$

Again from Gow's result, $\varepsilon(\phi_1) = 1$, and so there is a symmetric nondegenerate $\mathrm{O}(n, \mathbb{F}_q)$-invariant bilinear form B on V_1, unique up to scalar. Then if B_ι restricted to $V_1 \times V_1$ is nondegenerate, it would have to be a scalar multiple of B, and so B_ι would be symmetric on $V_1 \times V_1$. But since B_ι is either symmetric or skew-symmetric on all of $V \times V$, then being nondegenerate and symmetric on a subspace forces it to be symmetric everywhere. So now we must show B_ι is nondegenerate on $V_1 \times V_1$.

For $g \in \mathrm{O}(n, \mathbb{F}_q)$, $u \in V_1$, and $v \in V_2$, we have

$$B_\iota(\pi(g)u, {}^{\iota}\pi(g)v) = B_\iota(\phi_1(g)u, \phi_2(g)v) = B_\iota(u, v).$$

So if B_ι is nondegenerate on $V_1 \times V_2$, then we would have $\hat{\phi}_1 \cong \phi_2$. But $\phi_2 \cong \hat{\phi}_2$, and so we would have $\phi_2 \cong \phi_1$. This would imply that $\pi_1 \cong \pi_2$, since the central characters of π_1 and π_2 agree with the central character of π. But we cannot have π restricted to an index 2 subgroup be the direct sum of 2 isomorphic representations, by [5, Corollary 6.19]. So now B_ι must be zero on $V_1 \times V_2$, by Schur's Lemma, which means B_ι must be nondegenerate on $V_1 \times V_1$, since B_ι is nondegenerate on $V \times V$ and $V = V_1 \oplus V_2$. Therefore, B_ι is symmetric, and $\varepsilon_\iota(\pi) = 1$. ■

Kawanaka and Matsuyama [6] obtained a formula for the invariants $\varepsilon_\iota(\pi)$ which generalized the classical formula of Frobenius and Schur. One of the results in [6], which generalizes the Frobenius–Schur involution formula, is that if $\varepsilon_\iota(\pi) = 1$ for all irreducible representations π of a group G, then the sum of the degrees of the irreducibles of G is equal to the number of elements in G satisfying $g\,{}^{\iota}g = 1$. From this and Theorem 2, we obtain the following.

Corollary 2 *Let q be odd and let* $G = \mathrm{GO}(n, \mathbb{F}_q)$. *The sum of the degrees of the irreducible representations of G is equal to*

$$\left|\{g \in G \mid g^2 = \mu(g)I\}\right|.$$

It is perhaps worth noting that in the case of the group of similitudes for a split orthogonal group over $\mathbb{F}_q$, this is equal to the number of symmetric matrices in G.

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