Cartan Matrices with Null Roots and Finite Cartan Matrices

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A Cartan matrix is an $n \times n$ matrix (A_{ij}) with integer coefficients which satisfies the conditions

(i) $A_{ii} = 2, i = 1, 2, \cdots, n$,
(ii) $A_{ij} \leqq 0$ if $i \neq j$,
(iii) $A_{ij} = 0$ if and only if $A_{ji} = 0$.

We say that (A_{ij}) has a null root if there exists a non-zero column vector $[d_j] = [d_1, d_2, \cdots, d_n]$ such that $(A_{ij})[d_j] = 0$, where each d_i is a non-negative integer. We call (A_{ij}) symmetrizable if there exists a non-singular diagonal matrix D such that the product $(A_{ij})D$ is a symmetric matrix.

Nowadays it is customary to represent Cartan matrices by diagrams which are a slight modification of the diagrams introduced by Coxeter to classify the discrete groups generated by reflections (see [3] or [5] chapter XI). Attached to a Cartan matrix there exists a group generated by reflections called its Weyl group. This is defined in Section 3 as well as the root system connected with it. When the matrix is symmetrizable the Weyl group leaves invariant a non-trivial quadratic form.

Cartan matrices are used in the construction of Lie algebras $\mathfrak{L}(A_{ij})$ over fields of characteristic 0; they generalize the matrices used by E. Cartan in his thesis of 1894 to classify the finite dimensional simple Lie algebras over the complex field. The result is that the algebra $\mathfrak{L}(A_{ij})$ is simple if (A_{ij}) is indecomposable in the sense defined in Section 1, and has no null roots; and it is finite dimensional if and only if it is one of the matrices obtained by Cartan. Because of this, we refer to these matrices as the finite Cartan matrices. The simplicity of $\mathfrak{L}(A_{ij})$ was established by Moody in [7] for symmetrizable matrices without null roots and in the general case by Kac in [6]. The extension of the construction to any field is carried out in [1], where it is found that with minor restrictions on the characteristic the result holds.

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The symmetrizable Cartan matrices with null roots are determined in [7] using properties of $\mathfrak{L}(A_{ij})$ and the classification of the finite Cartan matrices. Kac simply lists their diagrams and refers to [3]. These diagrams are also included in [8] and [2] in relation with Lie algebras, and in [4] they show up again in connection with a different problem.

In the present paper, after introducing some notation, we determine in Section 2 all the indecomposable Cartan matrices with null roots. It is seen that the lists given by Moody and Kac are complete because all such matrices are symmetrizable. In Section 3 we use the diagrams of the Cartan matrices with null roots to determine in a new way the diagrams of the finite Cartan matrices. To avoid the theory of Lie algebras, we define them as those Cartan matrices with a finite root system. The paper is self-contained and we think that our method, besides giving more information in the theory of Lie algebras, is shorter and simpler than the known ones.

§1. Preliminaries. Diagrams with weighted arrows. We represent the $n \times n$ Cartan matrix (A_{ij}) by a diagram in the following way,

a) the diagram has n vertices,

b) for $i \neq j$ we draw $|A_{ij}|$ arrows from the vertex j to the vertex i. Each such arrow will be called a (j, i)-arrow.

c) To simplify the diagram, when $|A_{ij}| = |A_{ji}| = 1$ we simply draw a line from i to j. If $|A_{ij}| > 1$, but $|A_{ji}| = 1$, we omit the (i, j)-arrow. There is no danger of confusion, because, if there are (j, i)-arrows, then (iii) implies that there is at least one (i, j)-arrow. If no such arrow appears in the diagram, it means that there is just one, which has to be taken into account when we count arrows.

It is clear that the diagram determines the Cartan matrix up to a permutation of the indices.

A Cartan matrix is called indecomposable if the corresponding diagram is connected, *i.e.*, $\{1, 2, \cdots, n\}$ can not be split into two non-empty subsets S and T such that $A_{ij} = 0$ if $i \in S$ and $j \in T$.

We attach a positive rational number a_i to the vertex i of the diagram for $i = 1, 2, \cdots, n$. Then we give the weight a_j/a_i to each (j, i)-arrow.

Remark. Property iii) implies that if there is an arrow of weight x there also exists an arrow of weight $1/x$.

§2. Indecomposable Cartan matrices with null-roots. A null root is by definition a non-negative solution of the homogeneous system of linear equations:

$$\sum_{j=1}^{n} A_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n.$$

Because $A_{ii} = 2$ and $A_{ij} \leqq 0$, if $[d_1, d_2, \cdots, d_n]$ is a null root we have

$$\sum_{j \neq i} |A_{ij}|\, d_j = 2d_i\,, \qquad i = 1, \cdots, n.$$

Assume $d_i = 0$, then the i-th equation becomes $\sum_{j \neq i} |A_{ij}|\, d_j = 0$, which implies that $d_j = 0$ if $A_{ij} \neq 0$. Since the matrix is indecomposable we would get $d_k = 0$ for all k, which contradicts the definition of null root. Therefore the coefficients of a null root are all positive and any two null roots are linearly dependent. If we attach to the vertex i the number d_i, the weight of an arrow is independent of the chosen null root. Our problem is reduced to finding the connected weighted diagrams which satisfy the condition: *For each vertex i the sum of the weights of all (j, i)-arrows is equal to 2.*

From the remark at the end of Section 1 it follows that,

(*) if w is the weight of an arrow, then $\frac{1}{2} \leqq w \leqq 2$.

Hence there are at most 4 arrows arriving at a vertex.

Now (*) implies that, if the condition holds, there are no weights with value between $\frac{3}{2}$ and 2. Using the remark we see that

(**) there are no weights with values between $\frac{1}{2}$ and $\frac{2}{3}$.

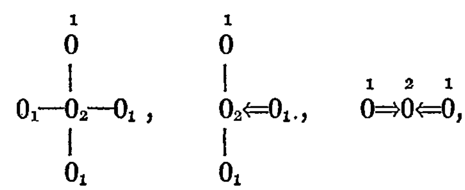
This, in turn, implies that there are no weights between $\frac{4}{3}$ and $\frac{3}{2}$. Hence, by the remark,

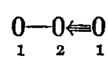
(***) there are no weights between $\frac{2}{3}$ and $\frac{3}{4}$.

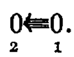
Our information about the weights shows that if there are 3 arrows arriving at a vertex, the only ways in which the weights can add up to 2 are:

$$\tfrac{1}{2} + \tfrac{1}{2} + 1; \quad \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{5}{6}; \quad \tfrac{1}{2} + \tfrac{3}{4} + \tfrac{3}{4} \text{ and } \tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3}.$$

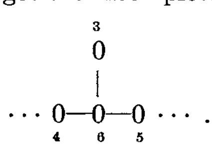
Suppose now that there are 4 arrows arriving at a vertex. Then each weight is $\frac{1}{2}$ and we get the following possibilities,



(1) 

(2) 

If there are 3 arrows arriving at a vertex, suppose we are in case $2 = \frac{1}{2} + \frac{2}{3} + \frac{5}{6} = \frac{3}{6} + \frac{4}{6} + \frac{5}{6}$, we get the incomplete diagram



The vertex with a 3 has an arrow of weight 2, so it is already saturated. As for the vertex with a 5 (4), there must be only one more arrow arriving at it with weight $\frac{4}{5}$ ($\frac{2}{4}$). Hence we obtain

```
            3
            0
            |
0—0—0—0—0 · · · .
2   4   6   5   4
```

Now the vertex with a 2 is already saturated but at the other end the vertex with a 4 needs an arrow with weight $\frac{3}{4}$. Proceeding in this way we arrive at

```
            3
            0
            |
0—0—0—0—0—0—0—0.
2   4   6   5   4   3   2   1
```
(3)

Let us take up now the expression $2 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$. Starting with

```
          ⋮
          0²
          |
· · · 0—0—0 · · ·
      2   3   2
```

we obtain

```
        0¹
        |
        0₂
        |
0—0—0—0—0.
1   2   3   2   1
```
(4)

But in this case we could also have

(5)
```
0—0—0⇐0—0
1   2   3   2   1
```

and

(6)
```
0⇐0—0.
3   2   1
```

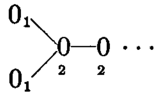
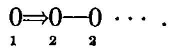
Similarly, from $2 = \frac{1}{2} + \frac{3}{4} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} + \frac{3}{4}$ we get

```
            0²
            |
0—0—0—0—0—0—0
1   2   3   4   3   2   1
```
(7)

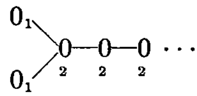
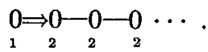
and

(8)
```
0—0—0⇒0—0.
1   2   3   4   2
```

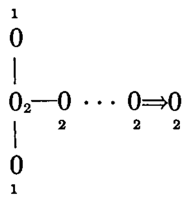
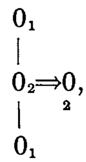
As for $2 = \frac{1}{2} + \frac{1}{2} + 1$ we start with

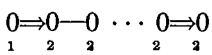
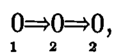
either  or  .

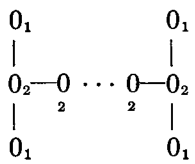
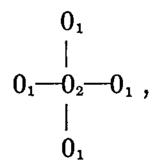
We can add a vertex and obtain

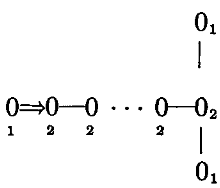
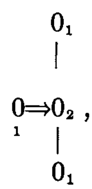
 or  .

The ways of completing such diagrams are,

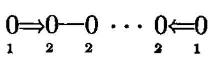
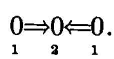
(9)  with the special case ,

(10)  with the special case , or

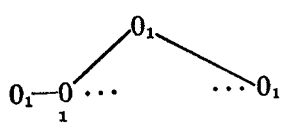
(11)  with the special case ,

(12)  with the special case ,

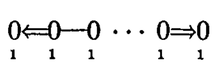
as well as,

(13)  with the special case .

All other diagrams will have at most 2 arrows arriving at a vertex. These are

(14) 

and

(15) 

with the special case

$$0 \underset{1}{} \overset{\Rightarrow}{\underset{\Leftarrow}{}} \underset{1}{0}. \tag{16}$$

Remark. If we allow infinitely many entries in our indecomposable Cartan matrices, there can be no null roots since we only consider vectors with a finite number of non-zero entries.

§3. **Finite Cartan matrices.** Let V_0 be a vector space over the rational field Q with basis $\alpha_0, \alpha_1, \cdots, \alpha_n$ and V the subspace spanned by $\alpha_1, \alpha_2, \cdots, \alpha_n$. Given an $n \times n$ Cartan matrix we define linear transformations S_i, S_i^*, $1 \leqq i \leqq n$, acting on V by $\alpha_j S_i = \alpha_j - A_{ij}\alpha_i$ and $\alpha_j S_i^* = \alpha_j - A_{ji}\alpha_i$, and introduce a pairing

$$(\cdot\ ,\ \cdot): V \times V \to Q$$

defined on our basis by $(\alpha_i, \alpha_j) = A_{ji}$. It is immediate that $\alpha S_i = \alpha - (\alpha, \alpha_i)\alpha_i$ and, $\alpha S_i^* = \alpha - (\alpha_i, \alpha)\alpha_i$, and it follows from this that S_i, S_i^* are reflections on V. That is, $S_i^2 = id = S_i^{*2}$ and S_i, S_i^* fix a hyperplane of V pointwise. Let W (respectively W^*) denote the group generated by the elements S_i (respectively S_i^*) for $1 \leqq i \leqq n$. W is called the Weyl group of (A_{ij}) so that W^* is the Weyl group of the Cartan matrix $(A_{ij})^t$ where t denotes transpose. Notice that $(\alpha S_K, \beta S_K^*) = (\alpha, \beta)$ for $\alpha, \beta \in V$ and hence by iteration $(\alpha S_{i_1} \cdots S_{i_r}, \beta S_{i_1}^* \cdots S_{i_r}^*) = (\alpha, \beta)$ for any $\alpha, \beta \in V$, $r \geqq 1$ and arbitrary indices $i_1, \cdots, i_r \in \{1, \cdots, n\}$.

Definition. Let (A_{ij}) be an $n \times n$ Cartan matrix and W its Weyl group. The elements of the set

$$\triangle = \{\alpha_i\omega \mid 1 \leqq i \leqq n, \quad \omega \in W\}$$

are called the *roots* of (A_{ij}), and $\triangle$ is called the *root system* of (A_{ij}). We are interested in those Cartan matrices for which $\triangle$ is finite, which we call finite Cartan matrices.

Lemma 1. *Let (A_{ij}) be a finite Cartan matrix. Then it is non-singular and symmetrizable with $D = \text{diag}(\epsilon_1, \cdots, \epsilon_n)$, all $\epsilon_i > 0$.*

Proof. Let $f(\cdot\ ,\ \cdot): V \times V \to Q$ be any positive definite symmetric bilinear form on V and define $\langle \cdot\ ,\ \cdot \rangle: V \times V \to Q$ by $\langle \alpha, \beta \rangle = \sum_{\omega \in W} f(\alpha\omega, \beta\omega)$ for any $\alpha, \beta \in V$. Here W denotes the Weyl group of (A_{ij}), which is finite, since $\triangle$ is finite and W can be faithfully represented as a group of permutations of $\triangle$. Clearly $\langle \cdot\ ,\ \cdot \rangle$ is a positive definite symmetric bilinear form on V for which $\langle \alpha\omega, \beta\omega \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in V$, $\omega \in W$. It follows from this, and the fact that S_i is a reflection on V, that $\alpha S_i = \alpha - 2(\langle \alpha, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle)\alpha_i$ for all $\alpha \in V$. In particular, taking $\alpha = \alpha_j$ we obtain that $A_{ij} = 2\langle \alpha_j, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle$. Thus, setting $\epsilon_j = 1/\langle \alpha_j, \alpha_j \rangle$, we have $\epsilon_j > 0$ and $A_{ij}\epsilon_j = A_{ji}\epsilon_i$. Since $(\langle \alpha_i, \alpha_j \rangle)$ is non-singular it is clear that (A_{ij}) is non-singular.

Lemma 2. *Let (A_{ij}) be an indecomposable symmetrizable Cartan matrix with $D = \text{diag}(\epsilon_1, \cdots, \epsilon_n)$. Then*

$$(\alpha_i S_{i_1} \cdots S_{i_r}, \alpha_j) = (\alpha_j, \alpha_i S_{i_1}^* \cdots S_{i_r}^*)\, \epsilon_j/\epsilon_i$$

for any $i, j, i_1, \cdots, i_r \in \{1, \cdots, n\}$.

Proof. We use induction on r, the result being clear if $r = 0$ because

$$A_{ij}\epsilon_j/\epsilon_i = (\alpha_j, \alpha_i)\epsilon_j/\epsilon_i .$$

Next suppose the result is true up to and including r and set $\alpha = \alpha_i S_{i_1} \cdots S_{i_r}$, $\alpha^* = \alpha_i S_{i_1}^* \cdots S_{i_r}^*$. We must show $(\alpha S_K, \alpha_j) = (\alpha_j, \alpha^* S_K^*)\epsilon_j/\epsilon_i$ for $1 \leqq K \leqq n$. Now $(\alpha S_K, \alpha_j) = (\alpha - (\alpha, \alpha_K)\alpha_K, \alpha_j) = (\alpha, \alpha_j) - (\alpha, \alpha_K)(\alpha_K, \alpha_j)$. By induction and the case $r = 0$ this equals

$$\frac{\epsilon_j}{\epsilon_i}(\alpha_j, \alpha^*) - \frac{\epsilon_K}{\epsilon_i}(\alpha_K, \alpha^*)\frac{\epsilon_j}{\epsilon_K}(\alpha_j, \alpha_K) = \frac{\epsilon_j}{\epsilon_i}(\alpha_j, \alpha^* - (\alpha_K, \alpha^*)\alpha_K)$$

$$= \frac{\epsilon_j}{\epsilon_i}(\alpha_j, \alpha^* S_K^*), \quad \text{as asserted.}$$

We now take an $n \times n$ indecomposable finite Cartan matrix (A_{ij}). If $0 \neq \alpha = \sum_{i=1}^n d_i\alpha_i \in V$ we define the level of α, denoted $\ell(\alpha)$, by $\ell(\alpha) = \sum_{i=1}^n d_i$. If all $d_i \geqq 0$ we say that α is positive. Since the root system Δ is finite, for a positive root δ of maximal level we get $\ell(\delta S_i) \leqq \ell(\delta)$ for $1 \leqq i \leqq n$. Fix a representation of δ, say, $\delta = \alpha_h S_{i_1} \cdots S_{i_r}$, and define $\delta^* \in \Delta^*$ by $\delta^* = \alpha_h S_{i_1}^* \cdots S_{i_r}^*$. Because $\ell(\delta S_j) \leqq \ell(\delta)$, and $\delta S_j = \delta - (\delta, \alpha_j)\alpha_j$, we have $(\delta, \alpha_j) \geqq 0$ for all j. By Lemma 2, $(\alpha_j, \delta^*) = (\delta, \alpha_j)\epsilon_h/\epsilon_j$; hence $(\alpha_j, \delta^*) \geqq 0$ and $(\alpha_j, \delta^*) = 0$ if and only if $(\delta, \alpha_j) = 0$. Also, $(\delta, \delta^*) = 2$, which implies, in particular, that not all (δ, α_j) are zero.

We now extend the pairing $(\cdot, \cdot): V \times V \to Q$ to a pairing $(\cdot, \cdot): V_0 \times V_0 \to Q$ by defining

$$(\alpha_i, \alpha_j) = \begin{cases} A_{ji} & \text{if } 1 \leqq i, j \leqq n, \\ 2 & \text{if } i = 0 = j, \\ -(\delta, \alpha_j) & \text{if } i = 0, 1 \leqq j \leqq n, \\ -(\alpha_i, \delta^*) & \text{if } 1 \leqq i \leqq n, j = 0. \end{cases}$$

Letting $B_{ji} = (\alpha_i, \alpha_j)$ we find from the above that (B_{ij}) is an indecomposable Cartan matrix. Moreover, $0 \neq \alpha_0 + \delta = \sum_{j=0}^n d_j\alpha_j$ for some non-negative $d_j \in Q$, by our choice of δ. Now $(\alpha_0 + \delta, \alpha_i) = -(\delta, \alpha_i) + (\delta, \alpha_i) = 0$ for $1 \leqq i \leqq n$, and $(\alpha_0 + \delta, \alpha_0) = 2 + (\delta, \alpha_0) = 2 - (\delta, \delta^*) = 0$, and hence

$$(B_{ij})\begin{pmatrix} d_0 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, (B_{ij}) is an indecomposable Cartan matrix with a null root.

By our construction of (B_{ij}) we find that the diagram of (A_{ij}) is obtained from that of (B_{ij}) by removing a vertex for which the remaining diagram is connected. Thus, by simply examining the sixteen diagrams obtained in Section 2 we find that (A_{ij}) can only correspond to one of the nine types of diagrams listed below.

A_l 0—0 ··· 0—0, $l \geqq 1$,

B_l 0⇐0—0 ··· 0—0, $l \geqq 2$,

C_l 0⇒0—0 ··· 0—0, $l \geqq 3$,

```
              0
              |
D_l  0—0 ··· 0—0,        l ≧ 4,

          0
          |
E_6  0—0—0—0—0,

              0
              |
E_7  0—0—0—0—0—0,

                  0
                  |
E_8  0—0—0—0—0—0—0.
```

F_4 0—0=0—0

G_2 0≡0.

Remark. It is well known that all of these diagrams actually do correspond to finite Cartan matrices (see [2] for instance). However, this is quite easy to see directly using some elementary but ingenious results on matrices in [5]. Indeed let $B = (B_{ij})$, $1 \leqq i, j \leqq \ell$ be one of the nine types listed above. Then so is $A \equiv B^t$. By adding a suitable node to A we can obtain a matrix $\tilde{A} = (A_{ij})$, $0 \leqq i, j \leqq \ell$ which is one of those found in Section 2. It has a null root $[d] = [d_0, \cdots, d_\ell]$ (just read the d_i's off the diagram). Let $\tilde{D} = \operatorname{diag}\{\epsilon_0, \epsilon_1, \cdots, \epsilon_\ell\}$ be a positive diagonal matrix such that $\tilde{A}\tilde{D}$ is symmetric (choose $\epsilon_0 > 0$ arbitrarily and the other ϵ_i are uniquely determined). Let $D = \operatorname{diag}\{\epsilon_1, \cdots, \epsilon_\ell\}$. Now $\tilde{A}\tilde{D}$ is a connected a-form in the sense of Coxeter [5] and $\tilde{A}\tilde{D}\ (\tilde{D}^{-1}[d]) = 0$. By [5] 10.33 and 10.24, $\tilde{A}\tilde{D}$ is positive semi-definite and $AD \equiv (a_{ij})$ is positive definite. Each S_i^*, $i = 1, \cdots, \ell$ is easily seen to be an isometry relative to the positive definite form f: $f(\alpha_i, \alpha_j) = a_{ij}$. Thus $\Delta^* \equiv \{\alpha_1, \cdots, \alpha_\ell\}W^*$ is contained in the bounded region $\{x: f(x, x) \leqq \max(a_{11}, \cdots, a_{\ell\ell})\}$ and also in the lattice $\mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_\ell$, whence Δ^* is finite. Since Δ^* is the root system of B, B is a finite Cartan matrix.

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