SOME EXAMPLES OF COMPLEMENTED MODULAR LATTICES

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Let L be a complemented, $\aleph$-complete modular lattice. A theorem of Amemiya and Halperin (see [1], Theorem 4.3) asserts that if the intervals $[O, a]$ and $[O, b]$, $a, b \in L$, are upper $\aleph$-continuous then $[O, a \cup b]$ is also upper $\aleph$-continuous. Roughly speaking, in L upper $\aleph$-continuity is additive. The following question arises naturally: is $\aleph$-completeness an additive property of complemented modular lattices? It follows from Corollary 1 to Theorem 1 below that the answer to this question is in the negative.

A complemented modular lattice is called a Von Neumann geometry or continuous geometry if it is complete and continuous. In particular a complete Boolean algebra is a Von Neumann geometry. In any case in a Von Neumann geometry the set of elements which possess a unique complement form a complete Boolean algebra. This Boolean algebra is called the centre of the Von Neumann geometry. Theorem 2 shows that any complete Boolean algebra can be the centre of a Von Neumann geometry with a homogeneous basis of order n (see [3] Part II, definition 3.2 for the definition of a homogeneous basis), n being any fixed natural integer.

Preliminaries

We first recall some properties of regular rings. The definitions and proofs can be found in [3] part II, Chap. II or [2], § 3. We always assume that the regular ring has a unit element which will be denoted by 1.

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If S is a regular ring, $\overline{L}_S$ $(\overline{R}_S)$ denotes the complemented modular lattices of principal left (right) ideals. The mapping which takes each element of $\overline{L}_S$ into its right annihilator is a dual-isomorphism of $\overline{L}_S$ onto $\overline{R}_S$. Under this map the principal left ideal $(e)_{\ell}$ generated by the idempotent e goes into the principal right ideal $(1-e)_r$.

If S is a regular ring, the ring S_n of $n \times n$ matrices with entries in S is also regular. There exists a lattice isomorphism between $\overline{L}_{S_n}$ $(\overline{R}_{S_n})$ and the lattice of finitely generated submodules of the left (right) S-module of n-tuples $(a_1, a_2, \ldots, a_n)$, $a_i \in S$. Since S_n is regular, for every $A \in S_n$ there exists an idempotent matrix E such that $(E)_{\ell} = (A)_{\ell}$. Moreover, it is possible to choose

$$E = \begin{pmatrix} e_1 & 0 & \ldots & 0 \\ c_{21} & e_2 & \ldots & 0 \\ . & . & \ldots & . \\ c_{n1} & c_{n2} & \ldots & e_n \end{pmatrix}$$

where $e_i^2 = e_i$, $e_i c_{ij} = c_{ij}$, $c_{ij} e_j = 0$, for $i, j = 1, 2, \ldots, n$ and $c_{ij} = 0$, for $j > i$. Such a matrix is called a left canonical matrix. An idempotent matrix such that $e_i^2 = e_i$, $c_{ij} e_j = c_{ij}$, $e_i c_{ij} = 0$ for $i, j = 1, 2, \ldots, n$ and $c_{ij} = 0$ for $j > i$ is called right canonical. For every $A \in S_n$ there exists a right canonical matrix E such that $(A)_r = (E)_r$. Notice that if E is a right (left) canonical matrix then $1-E$ is left (right) canonical.

In what follows our regular ring S will be the Boolean ring B defined by a Boolean algebra $\mathcal{B}$, that is, the elements of B are those of $\mathscr{B}$ and

$$a + b = ab' \cup ba' \ , \ ab = a \cap b \ ,$$

where c' denotes the complement of $c \in \mathscr{B}$. The notation $c = a \dot{\cup} b$ implies that $ab = 0$. If $\mathscr{I}$ is an ideal of $\mathscr{B}$, it defines an ideal I of B. There exists a 1-1 correspondence between the elements of $\mathscr{B}$ and the principal ideal of B.

In the ring S_n there is in general more than one left (right) canonical matrix corresponding to an element $A \in S_n$. However, if two left canonical matrices E and F are such that $(E)_{\mathscr{L}} = (F)_{\mathscr{L}}$ and they have the same idempotents down the main diagonal, then $E = F$. This follows from the fact that $EF = E$ if $(E)_{\mathscr{L}} = (F)_{\mathscr{L}}$. Although in general the element e_i is not uniquely defined by A, the ideal $(e_i)_{\mathscr{L}}$ is unique. Since in the Boolean ring B any principal ideal is defined by a unique element, any principal left ideal of B_n is defined by a unique left canonical matrix. We will identify the elements of $\overline{L}_{B_n}$ with the corresponding left canonical matrices.

Some examples of complemented modular lattices

Throughout this section $\mathscr{B}$ will be a Boolean algebra, $\mathscr{I}$ an ideal of $\mathscr{B}$, and B and I the corresponding Boolean ring and ideal. $\overline{J}$ denotes the cardinal power of the set J.

THEOREM 1. Let L consist of the 2 x 2 left canonical matrices

$$A = \begin{pmatrix} e_1 & 0 \\ a & e_2 \end{pmatrix}, \text{ where } e_1, e_2 \in B \text{ and } a \in I. \text{ For}$$

$A_1, A_2 \in L$, define $A_1 \leq A_2$ if $(A_1)_{\mathscr{L}} \subset (A_2)_{\mathscr{L}}$ where $(A)_{\mathscr{L}}$ is the principal left ideal of B_2 generated by A. Then L is a complemented modular lattice. Moreover, the following conditions are equivalent

(i) L is an $\aleph_\alpha$-complete $\aleph_\alpha$-sublattice of $\overline{L}_{B_2}$

(ii) L is an $\aleph_\alpha$-complete $\aleph_\alpha$-continuous $\aleph_\alpha$-sublattice of $\overline{L}_{B_2}$

(iii) $\mathscr{I}$ is an $\aleph_\alpha$-ideal and $\mathscr{B}$ is $\aleph_\alpha$-complete.

Proof. Let R be the set of right canonical matrices $A = \begin{pmatrix} e_1 & 0 \\ a & e_2 \end{pmatrix}$, $e_1, e_2 \varepsilon B$ with $a \varepsilon I$, ordered by the relation $A_1 \leq A_2$ if $(A_1)_r \subset (A_2)_r$. Then the dual isomorphism between $\overline{L}_{B_2}$ and $\overline{R}_{B_2}$ induces a dual isomorphism between L and R. Hence any statement about L implies its dual, since what we prove for L can be proved as well for R.

We show first that L is a complemented modular lattice. When $\mathscr{I} = \mathscr{B}$ the ordered set defined in the theorem coincides with $\overline{L}_{B_2}$ and there is nothing to prove. When $\mathscr{I} \neq \mathscr{B}$ we will prove that L is a sublattice of $\overline{L}_{B_2}$. For this we use the lattice isomorphism between the principal left ideals of B_2 and the finitely generated submodules of the left B-module of 2-tuples (a_1, a_2), $a_i \varepsilon B$. If $\{(a_1, a_2)\}$ denotes the left submodule generated by the vector (a_1, a_2) then the module M corresponding to the canonical matrix $\begin{pmatrix} e_1 & 0 \\ a & e_2 \end{pmatrix}$ has the form

(1) $M = \{(e_1, 0)\} \oplus \{(a, e_2)\} = \{(e_1, 0)\} \oplus \{(a, a)\} \oplus \{(0, a_o)\}$

where $a_o = e_2 a'$ and $\oplus$ indicates direct sum. Since the matrix is canonical $e_2 = a \dot{\cup} a_o$.

It is clear that the only elements of M whose second or first component is zero are the elements of the submodules $\{(e_1, 0)\}$ or $\{(0, a_o)\}$, respectively. The elements of M of the form (c, c) are the elements of $\{(a \dot{\cup} e_1 a_o \, , \, a \dot{\cup} e_1 a_o)\}$.

The module

$$(2) \qquad N = \{(f_1, 0)\} \oplus \{(b, b)\} \oplus \{(0, b_o)\} \, ,$$

where $b \varepsilon I$, $bf_1 = bb_o = 0$, corresponds to the canonical matrix

$$\begin{pmatrix} f_1 & 0 \\ b & f_2 \end{pmatrix} ,$$

where $f_2 = b \dot{\cup} b_o$. Now N contains M if and only if

$$e_1 \leq f_1 \, , \quad a_o \leq b_o \quad \text{and} \quad a \leq b \dot{\cup} f_1 b_o \, ,$$

or what is equivalent,

$$e_1 \leq f_1 \, , \; e_2 \leq f_2 \, , \; a \leq b \cup f_1 f_2 \quad \text{and} \quad a_o b = 0.$$

In general given two modules M and N defined by (1) and (2)

$$M \quad N = \{(e_1 \cup f_1, \; 0)\} + \{(a \cup b, \; a \cup b)\} + \{(0, \; a_o \cup b_o)\} =$$

$$= \{(e_1 \cup f_1 \cup ba_o \cup b_o a, \; 0)\} \oplus \{(c, c)\} \oplus \{(0, a_o \cup b_o \cup be_1 \cup af_1)\}$$

where $c = af_1' b_o' \cup e_1' a_o' b \leq a \cup b \, \varepsilon \, I$. Hence $M \cup N \varepsilon L$. By duality $M \cap N \varepsilon L$. Therefore L is a sublattice of a modular lattice and is itself modular. Since

$$M' = \{(e_1' a', \; 0)\} \oplus \{(0, \; a_o')\}$$

is a complement of the module M, L is a complemented modular lattice.

Our next step is to show that if $\mathscr{B}$ is $\aleph$-complete then $\bar{L}_{B_2}$ is $\aleph$-complete. It is sufficient to show that $\bar{L}_{B_2}$ is upper $\aleph$-complete, because the lower $\aleph$-completeness follows by duality.

$$\text{Let } A^{\beta} = \begin{pmatrix} e_1^{\beta} & 0 \\ a^{\beta} & e_2^{\beta} \end{pmatrix} \in \bar{L}_{B_2} \quad \text{for all } \beta \in J,$$

where $\bar{J} \leq \aleph$. It is immediate that if $\mathscr{B}$ is $\aleph$-complete, the union of the corresponding modules

$$M_{\beta} = \{(e_1^{\beta}, 0)\} \oplus \{(a^{\beta}, a^{\beta})\} \oplus \{(0, a_o^{\beta})\} \quad \text{where } a_o^{\beta} = e_2^{\beta}(a^{\beta})'$$

is the module

$$M = \{(\cup e_1^{\beta}, 0)\} + \{(\cup a^{\beta}, \cup a^{\beta})\} + \{(0, \cup a_o^{\beta})\}$$

which corresponds to the canonical matrix

$$(3) \qquad A = \begin{pmatrix} \cup e_1^{\beta} \cup ((\cup a^{\beta}) \cdot (\cup a_o^{\beta})) & 0 \\ d & |\cup a^{\beta}| \cup |\cup a_o^{\beta}| \end{pmatrix}$$

where $\quad d = (\cup a^{\beta}) \cdot (\cup e_1^{\beta} \cup ((\cup a^{\beta}) \cdot (\cup a_o^{\beta})))'$.

Now we are ready to prove the equivalence of conditions (i), (ii), (iii).

(i) implies (ii). This is a consequence of the additivity of upper $\aleph$-continuity in complemented $\aleph$-complete modular lattices. For, if

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the intervals $[0, X]$ and $[0, Y]$ are both isomorphic to $\mathscr{B}$;

hence $L = [0, X \cup Y]$ is upper $\aleph$-continuous. Using duality we get that L is $\aleph$-continuous.

(ii) implies (iii). Since $\mathcal{B}$ is isomorphic to the interval $[0, X]$, if L is $\aleph_\alpha$-complete then B is $\aleph_\alpha$-complete. Now let

$$C^\beta = \begin{pmatrix} 0 & 0 \\ a^\beta & a^\beta \end{pmatrix} \varepsilon\, L$$

for all $\beta \varepsilon J$ and $\overline{J} \leq \aleph_\alpha$. Then

$$\cup C^\beta = \begin{pmatrix} 0 & 0 \\ \cup a^\beta & \cup a^\beta \end{pmatrix} \varepsilon\, L ,$$

which implies that $\cup a^\beta \varepsilon \mathcal{A}$ and therefore $\mathcal{A}$ is $\aleph_\alpha$-complete.

(iii) implies (i). Let

$$A^\beta = \begin{pmatrix} e_1^\beta & 0 \\ a^\beta & e_2^\beta \end{pmatrix} \varepsilon\, L \text{ for all } \beta \varepsilon J ,$$

and $\overline{J} \leq \aleph_\alpha$. Then (3) implies that $\cup A^\beta \varepsilon L$, hence (i) holds.

COROLLARY 1. Let L be as in Theorem 1. Suppose $\mathcal{B}$ is complete and $\mathcal{A}$ is an $\aleph_\alpha$-ideal which is not an $\aleph_{\alpha+1}$-ideal. Then

(a) L contains two elements X and Y such that the intervals $[0, X]$ and $[0, Y]$ are complete and continuous and $L = [0, X \cup Y]$.

(b) L is $\aleph_\alpha$-complete and $\aleph_\alpha$-continuous but not $\aleph_{\alpha+1}$-complete.

Proof. The only thing which has to be proved is that L is not $\aleph_{\alpha+1}$-complete.

Suppose L is $\aleph_{\alpha+1}$-complete. Then, since $L = [0, X \cup Y]$, by the additivity of $\aleph_{\alpha+1}$-continuity in $\aleph_{\alpha+1}$-complete lattices, L is $\aleph_{\alpha+1}$-continuous. Let Ω be the first ordinal such that $\overline{\Omega} = \aleph_{\alpha+1}$ and $\{a^\beta\}_{\beta<\Omega}$ an increasing chain of elements of $\mathscr{I}$ such that $\cup a^\beta \notin \mathscr{I}$. Take

$$C^\beta = \begin{pmatrix} 0 & 0 \\ a^\beta & a^\beta \end{pmatrix}$$

Then

$$C = \cup C^\beta = \begin{pmatrix} 0 & 0 \\ \cup a^\beta & \cup a^\beta \end{pmatrix} \notin L .$$

If $C' = \begin{pmatrix} e_1 & 0 \\ b & * \end{pmatrix}$ is the supremum of the C^β in L then $b \not\geq \cup a^\beta$, since $b \in I$. On the other hand $C < C'$ implies that $\cup a^\beta \leq b \cup e_1$, hence $e_1 \neq 0$. Now $D = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \in L$. $D \cap C^\beta = 0$ for all $\beta < \Omega$, but $D \cap C \neq 0$, which contradicts the $\aleph_{\alpha+1}$-continuity of L.

COROLLARY 2. Let L be as in Theorem 1. Then L is a Von Neumann geometry if and only if $\mathscr{B}$ is a complete Boolean algebra and $\mathscr{I}$ is a principal ideal, that is, $I = [0, x]$, $x \in B$. In this case the center of L is isomorphic to $[0, x] \times [0, x'] \times [0, x']$.

Proof. When $\mathcal{A} = [0, x]$, L is the lattice direct sum of the intervals $[0, Y_o]$, $[0, Y_1]$, $[0, Y_2]$, where

$$Y_o = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad Y_1 = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & x' \end{pmatrix}.$$

Hence its center is isomorphic to $[0, x] \times [0, x\ ] \times [0, x\ ]$

THEOREM 2. If $\mathcal{B}$ is a complete Boolean algebra, then the lattice $\bar{L}_{B_n}$ is a Von Neumann geometry whose center is isomorphic to $\mathcal{B}$.

Remark. For $n = 2$ this theorem is contained in Theorem 1.

Proof. Because of the dual isomorphism between $\bar{L}_{B_n}$ and $\bar{R}_{B_n}$ we only need to prove that $\bar{L}_{B_n}$ is upper complete and upper continuous. Now $\bar{L}_{B_n} = [0, X_1 \cup X_2 \cup \ldots \cup X_n]$, where X_i is the canonical matrix with 1 in the (i, i) place and zeros elsewhere, and the interval $[0, X_i]$ being isomorphic to $\mathcal{B}$, is continuous. Therefore, by the theorem of Amemiya and Halperin quoted in the introduction, if $\bar{L}_{B_n}$ is upper complete it is also upper continuous. So it is sufficient to prove that $\bar{L}_{B_n}$ is upper complete.

We use induction on n. If $n=1$, $\bar{L}_B \approx \mathcal{B}$ and there is nothing to prove. Assume then that the theorem is true for $n-1$. Let A^β be an increasing chain, where $\beta < \Omega$, Ω any limit ordinal, and

$$E = \begin{pmatrix} 1 & 0 \ldots 0 & 0 \\ 0 & 1 \ldots 0 & 0 \\ . & . \ldots . & . \\ 0 & 0 \ldots 1 & 0 \\ 0 & 0 \ldots 0 & 0 \end{pmatrix} \quad \varepsilon\ \bar{L}_{B_n}.$$

Then the elements $A^{\beta} \cap E$ form an increasing chain. To the element $A^{\beta} \cap E$ there corresponds a finitely generated submodule N^{β} of the left B-module of n-tuples and the elements of this submodule have the last component equal zero. Therefore, because of the induction assumption, the increasing chain of submodules N^{β} has a supremum which is also a submodule whose elements have the last component equal to zero. Let $A' \varepsilon \bar{L}_{B_n}$ be the left canonical matrix corresponding to this submodule,

$$A' = \begin{pmatrix} e_1 & 0 & \dots 0 & 0 \\ c_{21} & e_2 & \dots 0 & 0 \\ . & . & \dots . & . \\ c_{n-1,1} & c_{n-1,2} & \dots e_{n-1} & 0 \\ 0 & 0 & \dots 0 & 0 \end{pmatrix}$$

If C is an upper bound of the A^{β}, $\beta < \Omega$, then $C \geq A^{\beta} \cap E$. Hence $C \geq A'$, and $C \geq A^{\beta} \cup A'$. That is, any upper bound of the A^{β} is an upper bound of the chain of $A^{\beta} \cup A'$ and conversely. Let $B^{\beta} = A^{\beta} \cup A'$, since $B^{\beta} \cap E = (A^{\beta} \cup A') \cap E = A'$,

$$B^{\beta} = \begin{pmatrix} e_1 & 0 & \dots 0 & 0 \\ c_{21} & e_2 & \dots 0 & 0 \\ . & . & \dots . & . \\ c_{n-1,1} & c_{n-1,2} & \dots e_{n-1} & 0 \\ b_1^{\beta} & b_2^{\beta} & \dots b_{n-1}^{\beta} & e_n^{\beta} \end{pmatrix}$$

Moreover, if $\alpha < \beta$, $B^{\alpha} \leq B^{\beta}$ and this implies $B^{\alpha}B^{\beta} = B^{\alpha}$, which is equivalent to $e_n^{\alpha} b_i^{\beta} = b_i^{\alpha}$, $i = 1, 2, \dots, n-1$, $e_n^{\alpha} e_n^{\beta} = e_n^{\alpha}$. Now it is easily seen that

$$B = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0 \\ c_{21} & e_2 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cup b_1^\beta & \cup b_2^\beta & \dots & \cup b_{n-1}^\beta & \cup e_n^\beta \end{pmatrix}$$

is the supremum of the chain of B^α. For, $e_n^\alpha b_i^\beta = b_i^\alpha$ and $e_n^\alpha e_n^\beta = e_n^\alpha$ for $\alpha < \beta$ imply that the b_i^β and e_n^β form increasing chains. Consequently, $e_n^\alpha(\cup b_i^\beta) = b_i^\alpha$, $e_n^\alpha(\cup e_n^\beta) = e_n^\alpha$ and $(\cup e_n^\alpha)(\cup b_i^\beta) = \cup_\alpha (e_n^\alpha(\cup_\beta b_i^\beta)) = \cup b_i^\alpha$. Therefore B is a canonical matrix such that $B^\alpha B = B^\alpha$, which implies $B^\alpha \leq B$, and it is clear that B is the supremum.

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